STABILITY OF THE EXPONENTIAL UTILITY MAXIMIZATION PROBLEM WITH RESPECT TO PREFERENCES

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Abstract. This paper studies stability of the exponential utility maximization when there are small variations on agent's utility. Two settings are studied. First, in a general semimartingale model where random endowments are present, there is a sequence of utilities defined on $\mathbb{R}$ converging to the exponential utility. Under a uniform condition on their marginal utilities, convergence of value functions, optimal terminal wealth and optimal investment strategies are obtained, their rate of convergence are determined. Stability of utility-based pricing is also discussed. Second, there is a sequence of utilities defined on $\mathbb{R}_+$ each of which is comparable to a power utility whose relative risk aversion converges to ∞. Their associated optimal strategies, after appropriate scaling, converge to the optimal strategy for the exponential hedging problem. This complements Theorem 3.2 in *M. Nutz, Probab. Theory Relat. Fields, 152, 2012*, by allowing general utilities in the converging sequence.

0. Introduction

This paper considers an optimal investment problem where an agent, whose preference is described by a utility function, seeks to maximize expected utility of her wealth from investment and a random endowment (illiquid asset) at an investment horizon $T \in \mathbb{R}_+$. Given two problem primitives: utility function and market structures, the goal is to identify the optimal investment strategy that the agent undertakes. When the utility has constant absolute risk aversion, Delbaen et al. (2002) give an elegant solution to the aforementioned problem. We study in this paper stability of the optimal investment strategy when agent's utility deviates from the exponential utility. In particular, we are interested in a *quantitative* measure on how far the optimal strategy deviates when there are small variations in agent's preference.

We consider two settings for the stability problem. First, consider a sequence of utility functions $(U_\delta)_{\delta>0}$, each of which is defined on $\mathbb{R}$, such that it converges pointwise to U_0 which has unit absolute risk aversion. The deviation of U_δ from U_0 is measured by two components: i) the ratio of marginal utilities $\mathfrak{R}_\delta$ between U_δ and an exponential utility $\widetilde{U}_\delta$ whose absolute risk aversion is α_δ; ii) the deviation of α_δ from 1. The first component measures how far U_δ is away from an exponential utility; while the second component describes how far the absolute risk aversion of the exponential utility is away from 1. When $\mathfrak{R}_\delta$ is bounded from above and away from zero, uniformly in δ, our first main result, Theorem 1.8, states the convergence in expectation of the

Date: March 5, 2013.

Key words and phrases. utility maximization, exponential utility, stability, semimartingales, utility-based prices.

We would like to thank Constantinos Kardaras, Johannes Muhle-Karbe, and Marcel Nutz for helpful comments. This research is supported in part by STICERD at London School of Economics.

optimal terminal wealth and the convergence of the value function in a general semimartingale model; moreover the convergence of the optimal strategy also holds, when asset price processes are continuous. Beyond these continuity results, the rate of convergence is determined in Corollary 1.10. Those two components on variation of utilities translate to impact on deviation of terminal wealth (hence optimal strategy) at different rate: *first order* on the convergence rate of the absolute risk aversion and *second order* on the convergence rate of $\mathfrak{R}_\delta$. Stability of utility based pricing, Davis price and indifference price, with respect to agent's preference is also discussed in the first setting; see Corollaries 1.12 and 1.13.

The stability problem studied in the first setting is similar to Carassus and Rásonyi (2007), where the problem is formulated in a discrete time setting and asset price processes are assumed to be bounded. For utilities defined on $\mathbb{R}_+$, aforementioned stability problem has been extensively studied. Jouini and Napp (2004) consider an Itô process model. Larsen (2009) extends the analysis to continuous semimartingale models. Kardaras and Žitković (2011) allows simultaneous variations of preferences and subjective probabilities. More recently, Mocha and Westray (2011) focus on power utility maximization problem and investigate stability respect to the relative risk aversion, the market price of risk, and investment constraints.

In Larsen (2009) and Kardaras and Žitković (2011), convergence in probability of the optimal terminal wealth is obtained under an uniform integrability assumption. After changing to a measure where the limiting optimal wealth process has the *numéraire property*, the optimal investment strategy also converges utilizing (Kardaras, 2010, Theorem 2.5). Our uniform bound on the ratio of marginal utilities implies an analogous integrability condition for utilities defined on $\mathbb{R}$; see Remark 1.9. However additional structure imposed allows us to obtain more precise infirmation on how fast the convergence takes place.

A different type of stability problem is studied in Larsen and Žitković (2007). Therein stability of the optimal terminal wealth with respect to market variations is studied while a utility defined on $\mathbb{R}_+$ is fixed. This type of stability problem has recently been investigated in Frei (2011) and Bayraktar and Kravitz (2011) for the exponential utility maximization problem.

In the second setting, we consider a sequence of utility random fields $(\mathcal{U}_p)_{p<0}$, each of which is of the form $\mathcal{U}_p = D\, U_p$ for a positive process D and a utility function U_p defined on $\mathbb{R}_+$. For each U_p, the ratio of its marginal utility with respect to x^{p-1} is bounded from above and away from zero. In this sense U_p is comparable to power utility with constant relative risk aversion $1-p$. As the relative risk aversion going to infinity and the ratio of marginal utilities going to 1, $(U_p)_{p<0}$ is closer to a sequence of power utilities $\widetilde{U}_p$ (see Remark 1.20) which converges to the exponential utility with appropriate domain shift (see (Nutz, 2012, Remark 3.3)).

Our second main result, Theorem 1.19, states that, when the ratio of marginal utilities converges to 1 at a rate at least as fast as the relative risk aversion going to infinity, then the optimal proportion invested in risky assets, scaled by $1-p$, converges to the optimal monetary value invested in risky assets in the exponential hedging problem. Therein $(1-p)^{-1}$ can be regarded as the rate of convergence. This result is first obtained in Nutz (2012) where U_p is power utility. We complement

Nutz's result by allowing deviation from power utility and analyze the impact on the convergence from the ratio of marginal utilities. On the dual side, the stability problem formulated here is related to the convergence of optimal martingale measures which is studied in Grandits and Rheinländer (2002), Mania and Tevzadze (2003), and Santacroce (2005).

The starting point of our proofs in both settings is the following key result from the *duality theory*: the optimal wealth process is a *martingale* after multiplied by the optimal dual process and a *supermartingale* after multiplied by any other process in the dual domain. When random endowment presents, aforementioned properties have been proved in Owen and Žitković (2009) for utility defined on $\mathbb{R}$ and in Karatzas and Žitković (2003) for utility defined on $\mathbb{R}_+$. These properties, combined with scaling properties of exponential (resp. power) utility, lead to an estimate on the difference (resp. ratio) of optimal terminal wealth between problems with U_δ (resp. U_p) and exponential (resp. power) utility. The remaining proof does not depend on the market specifications. Therefore methods in this paper could potentially be applied to other market settings where the aforementioned property on the optimal wealth process holds, for example, markets with transaction cost, see Cvitanić and Karatzas (1996), and the utility maximization with forward criteria, see Musiela and Zariphopoulou (2009). We leave this as a future research project.

The structure of the paper is simple. After this introduction, Section 1 describes the problems and states main results, while all proofs are given in Sections 2 and 3.

1. Main results

We consider a financial market of d-risky assets whose discounted prices are modeled by a locally bounded $\mathbb{R}^d$-valued semimartingale $(S_t)_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, in which $\mathcal{F}_0$ coincides with the family of $\mathbb{P}$-null sets and $(\mathcal{F}_t)_{t\in[0,T]}$ is right continuous. When price processes are non-locally bounded semimartingale, we refer the reader to (Biagini and Frittelli, 2005, 2007).

1.1. Utilities defined on $\mathbb{R}$.

In our first main result, we consider a sequence of utility functions $U_\delta : \mathbb{R} \to \mathbb{R}$, indexed by $\delta \geq 0$, each of which is strictly increasing, strictly concave, and continuously differentiable. This sequence of utilities is convergent in the following sense.

Assumption 1.1. $\lim_{\delta\downarrow 0} U_\delta(x) = U_0(x)$ for $x \in \mathbb{R}$, where $U_0(x) = -\exp(-x)$.

Here U_0 has unit absolute risk aversion. After appropriate scaling all results in this paper hold when U_0 has other value of absolute risk aversion. The pointwise convergence of utility functions in the previous assumption is widely used in the literature; see e.g. Jouini and Napp (2004) and Larsen (2009). The pointwise convergence, restricted to the class of concave functions (utility functions), implies a more economic meaningful mode of convergence: the pointwise (and hence locally uniformly) convergence of their derivatives (marginal utilities); see (Rockafellar, 1970, pp. 90 and pp. 248). We furthermore restrict each U_δ to a class of utilities which are comparable to the exponential utility $-\frac{1}{\alpha_\delta}\exp(-\alpha_\delta x)$.

Assumption 1.2. There exist strictly positive constants $\underline{\mathfrak{R}} \leq 1 \leq \overline{\mathfrak{R}}$, and $(\alpha_\delta)_{\delta>0}$ with $\lim_{\delta\downarrow 0} \alpha_\delta = 1$ such that

$$\underline{\mathfrak{R}} \leq \mathfrak{R}_\delta(x) := \frac{U'_\delta(x)}{\exp(-\alpha_\delta x)} \leq \overline{\mathfrak{R}}, \qquad \text{for all } \delta > 0 \text{ and } x \in \mathbb{R}.$$

Remark 1.3. This assumption implies that each U_δ is bounded from above. Indeed, integrating $\underline{\mathfrak{R}} \exp(-\alpha_\delta x) \leq U'_\delta(x) \leq \overline{\mathfrak{R}} \exp(-\alpha_\delta x)$ on $(0,\infty)$ yields $\underline{\mathfrak{R}}/\alpha_\delta + U_\delta(0) \leq U_\delta(\infty) \leq \overline{\mathfrak{R}}/\alpha_\delta + U_\delta(0)$. These bounds can be made uniform in δ, since $\lim_{\delta\downarrow 0} U_\delta(0) = -1$ and $\lim_{\delta\downarrow 0} \alpha_\delta = 1$. Moreover, U_δ is sandwiched between two utilities with constant absolute risk aversion α_δ. This is because integrating the previous bounds for $U'_\delta(x)$ on (x,∞) induces $U_\delta(\infty) - \frac{1}{\alpha_\delta}\overline{\mathfrak{R}} \exp(-\alpha_\delta x) \leq U_\delta(x) \leq U_\delta(\infty) - \frac{1}{\alpha_\delta}\underline{\mathfrak{R}} \exp(-\alpha_\delta x)$ for any $x \in \mathbb{R}$. One can also derive from the previous assumption that each U_δ satisfies the Inada conditions, i.e., $\lim_{x\downarrow -\infty} U'_\delta(x) = \infty$ and $\lim_{x\uparrow\infty} U'_\delta(x) = 0$, and U_δ has *reasonable asymptotic elasticity*, i.e.,

$$AE_{-\infty}(U_\delta) := \liminf_{x\downarrow -\infty} \frac{xU'_\delta(x)}{U_\delta(x)} > 1 \quad \text{ and } \quad AE_{\infty}(U_\delta) := \limsup_{x\uparrow\infty} \frac{xU'_\delta(x)}{U(x)} < 1.$$

Hence each U_δ is reasonable risk averse at high and low wealth limit; see (Kramkov and Schachermayer, 1999, 2003).

To introduce the utility maximization problem considered, we denote by M^a (resp. M^e) the class of probability measures $\widetilde{\mathbb{P}} \ll \mathbb{P}$ (resp. $\widetilde{\mathbb{P}} \sim \mathbb{P}$) such that S is a local martingale under $\widetilde{\mathbb{P}}$. Consider the convex conjugate $V_\delta : (0,\infty) \to \mathbb{R}$ defined by $V_\delta(y) := \sup_{x\in\mathbb{R}}(U_\delta(x) - xy)$. The *generalized entropy* of $\widetilde{\mathbb{P}} \in M^a$ relative to $\mathbb{P}$ and V_δ is defined as $\mathbb{E}_{\mathbb{P}}[V_\delta(d\widetilde{\mathbb{P}}/d\mathbb{P})] \in (0,\infty]$. We denote by $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) the set of probability measures $\widetilde{\mathbb{P}} \in M^a$ (resp. $\widetilde{\mathbb{P}} \in M^e$) with finite generalized entropy relative to $\mathbb{P}$ and V_δ. Even though definition of $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) depends on V_δ, Lemma 2.1 below shows that $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) are the same for all $\delta \geq 0$ under Assumption 1.2. Henceforth we drop the subscript δ to write $\mathcal{M}^a$ (resp. $\mathcal{M}^e$) instead.

There is an agent whose preference is described by one of the utility function U_δ. She is able to trade in the financial market and has a random endowment ξ_δ which is an $\mathcal{F}$-measurable random variable. Following Owen and Žitković (2009), we assume that ξ_δ is potentially unbounded but can be super-hedged.

Assumption 1.4. There exists $x_\delta, \widetilde{x}_\delta \in \mathbb{R}$ and a $\mathbb{P}$-predictable S-integrable process G_δ such that

$$x_\delta \leq \xi_\delta \leq \widetilde{x}_\delta + (G_\delta \cdot S)_T, \qquad \text{for each } \delta \geq 0,$$

where $G_\delta \cdot S$ is $\mathbb{P}$-a.s. uniformly bounded from below by a constant.

When the utility function is defined on $\mathbb{R}$, the class of uniformly bounded from below wealth processes is not large enough for the problem considered below; see e.g. Schachermayer (2001), therefore we recall the following class of permissible strategies from Owen and Žitković (2009): H is a *permissible trading strategy* if it is inside

$$\mathcal{H}^{perm} := \left\{ H : \begin{array}{l} H \text{ is a } \mathbb{P}\text{-predictable}, S\text{-integrable process such that} \\ H \cdot S \text{ is a } \widetilde{\mathbb{P}}\text{-supermartingale for all } \widetilde{\mathbb{P}} \in \mathcal{M}^a \end{array} \right\}.$$

Since $\mathcal{M}^a$ is the same for different δ, the class of permissible trading strategy defined above is independent of δ as well. Therefore even though the utility of the agent may change with respect to δ, she always choose trading strategy from the same permissible class.

Our agent chooses permissible strategies to maximize her utility on wealth and endowment at an investment horizon T:

$$u_\delta := \sup_{H\in\mathcal{H}^{perm}} \mathbb{E}_\mathbb{P}\left[U_\delta\left((H\cdot S)_T + \xi_\delta\right)\right]. \tag{1.1}$$

In order to ensure the existence of the optimal strategy, we impose

Assumption 1.5. $\mathcal{M}^e \neq \emptyset$.

When U_δ has reasonable asymptotic elasticity, $\mathcal{M}^a \neq \emptyset$, and Assumption 1.4 holds, Assumption 1.5 is actually the necessary and sufficient condition for the existence of optimal strategy for (1.1); see (Owen and Žitković, 2009, Theorem 1.9). We further recall the following result from Owen and Žitković (2009).

Proposition 1.6 (Owen-Žitković). *Let U_δ be of reasonable asymptotic elastic and Assumptions 1.4 and 1.5 hold. Then there exists an optimal strategy $H_\delta \in \mathcal{H}^{perm}$ for (1.1) such that $H_\delta\cdot S$ is a $\widetilde{\mathbb{P}}$-supermartingale for all $\widetilde{\mathbb{P}} \in \mathcal{M}^a$ and a $\mathbb{Q}_\delta$-martingale for some $\mathbb{Q}_\delta \in \mathcal{M}^e$, whose density $\mathbb{Q}_\delta$ satisfies*

$$y_\delta \frac{d\mathbb{Q}_\delta}{d\mathbb{P}} = U'_\delta\left((H_\delta\cdot S)_T + \xi_\delta\right), \qquad \text{for some positive constant } y_\delta.$$

In the previous result, $\mathbb{Q}_0$ is the the *minimal entropy measure* which minimizes $\mathbb{E}_\mathbb{P}[V_0(d\widetilde{\mathbb{P}}/d\mathbb{P})]$ among all $\widetilde{\mathbb{P}} \in M^a$. To simplify notation we drop the subscript 0 and denote the minimal entropy measure by $\mathbb{Q}$. In order to investigate the convergence of (1.1) and its optimal strategy as $\delta\downarrow 0$. We assume the following convergence of random endowment.

Assumption 1.7. There exists a constant $C\in\mathbb{R}_+$ such that $\alpha_\delta\xi_\delta - \xi_0 \geq -C$ $\mathbb{P}$-a.s. for all $\delta>0$. Moreover $\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}[|\alpha_\delta\xi_\delta - \xi_0|] = 0$.

The previous assumption clearly holds when ξ_δ is bounded uniformly in δ and $\mathbb{Q}-\lim_{\delta\downarrow 0}\xi_\delta = \xi_0$, where $\mathbb{Q}-\lim$ represents convergence in probability $\mathbb{Q}$. Denote the optimal payoff from investment by $X^\delta_T = (H_\delta\cdot S)_T$ for $\delta\geq 0$. The first main result states the convergence of X^δ_T, its associated strategy, and u_δ, as $\delta\downarrow 0$.

Theorem 1.8. *Let Assumptions 1.1-1.5 and 1.7 hold. Then the following statements hold:*

i) $\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}[|X^\delta_T - X^0_T|] = 0$;
ii) $\lim_{\delta\downarrow 0} u_\delta = u_0$;
iii) *If S is continuous then*

$$\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\left(\int_0^T (H_\delta - H_0)^\top_t d\langle S\rangle_t (H_\delta - H_0)_t\right)^{p/2}\right] = 0, \qquad \text{for any } p\in(0,1).$$

Remark 1.9. The analogue of Theorem 1.8, when $(U_\delta)_{\delta\geq 0}$ are defined on $\mathbb{R}_+$, has been proved in Larsen (2009) and Kardaras and Žitković (2011). Therein $\mathbb{P}-\lim_{\delta\downarrow 0} X^\delta_T/X^0_T = 1$ and $\lim_{\delta\downarrow 0} u_\delta = u_0$ are proved. Define $\overline{\mathbb{P}}$ via $d\overline{\mathbb{P}}/d\mathbb{P} = cU_0'(X^0_T)X^0_T$ for a normalization constant c. Then X^0 has the numéraire property under $\overline{\mathbb{P}}$, hence X^δ/X^0 is a $\overline{\mathbb{P}}$-supermartingale. Then $\lim_{\delta\downarrow 0}\mathbb{E}_{\overline{\mathbb{P}}}[|X^\delta_T/X^0_T - 1|] = 0$ and the convergence of the associated strategies follow from (Kardaras, 2010, Theorem 2.5).

Remark 1.3 implies that $(U_\delta)_{\delta>0}$ is uniformly bounded from above by

$$U_*(x) := \frac{\overline{\mathfrak{R}}}{\alpha_*} + \frac{\mathfrak{R}}{\alpha_*}\left(1-\exp(-\alpha_* x)\right), \qquad \text{where } \alpha_* = \min_{\delta>0}\alpha_\delta.$$

Since $\mathbb{E}_{\mathbb{P}}[V_*(d\mathbb{Q}/d\mathbb{P})] < \infty$, where V_* is the convex conjugate of U_*, $\{V_\delta(d\mathbb{Q}/d\mathbb{P})\}_{\delta>0}$ is clearly uniformly integrable under $\mathbb{P}$. This is the analogue of the uniform integrability assumption in Larsen (2009) and Kardaras and Žitković (2011). However the additional structure in Assumption 1.2 allows us to discuss the rate of convergence in what follows.

Define

$$f(\delta) := \sup_{x\in\mathbb{R}}|\mathfrak{R}_\delta(x) - 1| \qquad \text{and} \qquad g(\delta) := |\alpha_\delta - 1|, \qquad \text{for } \delta \geq 0.$$

They describe the rate of convergence for the ratio of marginal utilities and the absolute risk aversion.

Corollary 1.10. *Let Assumptions 1.1 and 1.5 hold. Suppose that* $\lim_{\delta\downarrow 0} f(\delta) = \lim_{\delta\downarrow 0} g(\delta) = 0$ *and* $\xi_\delta = x_0 \in \mathbb{R}$ *for all* $\delta \geq 0$. *Then*

$$\mathbb{E}_{\mathbb{Q}}\left[|X^\delta_T - X^0_T|\right] \sim O\left(f(\delta)^2 + g(\delta)\right), \qquad \textit{for sufficiently small } \delta.$$

Remark 1.11. When $U_\delta(x) := \frac{1}{\alpha_\delta}\exp(-\alpha_\delta x)$, it is clear that $X^\delta_T = X^0_T/\alpha_\delta$. Hence the optimal payoff X^δ_T converges to X^0_T at the rate of $g(\delta)$. When U_δ deviates from the exponential utility, the rate of convergence for the optimal payoff is determined by two components: convergence of the absolute risk aversion and the convergence of the ratio of marginal utilities. Corollary 1.10 shows that the rate of convergence is at least first order on the first component and second order on the second component. This provides a quantitative measure on how far X^δ_T is away from X^0_T.

When S is continuous, the previous result and the Burkholder-Davis-Gundy inequality combined imply that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (H_\delta - H_0)^\top_t d\langle S\rangle_t (H_\delta - H_0)_t\right)^{p/2}\right] \sim O\left(f(\delta)^2 + g(\delta)\right), \qquad \text{for any } p\in(0,1) \text{ and small } \delta,$$

see Lemma 2.4 and Corollary 2.5 for more details.

Another application of Theorem 1.8 is the stability of utility-based prices with respect to agent's preference. Consider a contingent claim B which satisfies

$$0 \leq B \leq \widetilde{x} + (G\cdot S)_T, \tag{1.2}$$

for some $\tilde{x}\in\mathbb{R}_+$ and a $\mathbb{P}$-predictable S-integrable process G such that $G\cdot S$ is $\mathbb{P}$-a.s. uniformly bounded from below by zero.

Let us first recall the *fair price* (Davis price) introduced in Davis (1997). An agent, endowed with utility U_δ and endowment ξ_δ, takes her preference into account to price the claim B as

$$\mathbb{E}_{\mathbb{Q}_\delta}[B],$$

where $\mathbb{Q}_\delta$ is introduced in Proposition 1.6. Theorem 1.8 allows us to establish the continuity of Davis price with respect to agent's preference.

Corollary 1.12. *Let Assumptions 1.1-1.5 and 1.7 hold. Then*

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}_\delta}[B]=\mathbb{E}_{\mathbb{Q}}[B].$$

Another utility-based pricing is the *indifference price* introduced into mathematical finance by Hodges and Neuberger (1989). See Carmona (2009) and references therein for recent development on this topic. Given an agent endowed with utility U_δ and initial wealth $x_0\in\mathbb{R}$, her *indifference buyer's price*, $p_\delta=p(B,x,U_\delta)$, of B is defined as the solution to the equation

$$u_\delta(x_0+B-p_\delta)=u_\delta(x_0),$$

where $u_\delta(\zeta)$ is defined in (1.1) with $\xi_\delta=\zeta$. The existence and uniqueness of p_δ is proved in (Owen and Žitković, 2009, Proposition 7.2). Theorem 1.8 ii) allows us to establish the following stability property of the indifference buyer's price with respect to agent's preference.

Corollary 1.13. *Let Assumptions 1.1, 1.2, and 1.5 hold. Suppose that either of the following conditions hold,*

i) *B satisfies* (1.2) *and $\alpha_\delta=1$ for all δ;*
ii) *B is bounded.*

Then $\lim_{\delta\downarrow 0}p_\delta=p_0$.

Remark 1.14. The continuity of Davis prices and indifference prices with respect to agent's preference has been investigated in Carassus and Rásonyi (2007) in a discrete time market with bounded stock price processes and bounded claims.

1.2. **Utilities defined on $\mathbb{R}_+$.** We continue with our second main result, which concerns the convergence of problems with utilities defined on $\mathbb{R}_+$ to the exponential utility maximization problem. Consider a sequence of utility random field $\mathcal{U}_p:[0,T]\times\Omega\times\mathbb{R}_+\to\mathbb{R}$ indexed by $p<0$. Each $\mathcal{U}_p$ is of the form

$$\mathcal{U}_p(t,x)=D_t\,U_p(x),\quad t\in[0,T],x\in\mathbb{R}_+,$$

where D is a càdlàg adapted positive process, $U_p:\mathbb{R}_+\to\mathbb{R}$ is strictly increasing, strictly concave, and continuously differentiable. We assume that each U_p is comparable to power utility x^p/p in the following sense.

Assumption 1.15. There exist strictly positive constants $\underline{\mathfrak{R}}_p\le 1\le\overline{\mathfrak{R}}_p$ such that

$$\underline{\mathfrak{R}}_p\le\mathfrak{R}_p(x):=\frac{U_p'(x)}{x^{p-1}}\le\overline{\mathfrak{R}}_p,\quad \text{for all } x\in\mathbb{R}_+.$$

Remark 1.16. The previous assumption implies that each U_p is bounded from above. Indeed, integrating $U_p'(x) \leq \overline{\mathfrak{R}}_p\, x^{p-1}$ on $(1, x)$ yields $U_p(x) \leq U_p(1) + \overline{\mathfrak{R}}_p(x^p/p - 1/p) \leq U_p(1) - 1/p\,\overline{\mathfrak{R}}_p$ for $x \geq 1$. Moreover U_p is sandwiched between two utilities with relative risk aversion $1-p$. Integrating $\underline{\mathfrak{R}}_p x^{p-1} \leq U_p'(x) \leq \overline{\mathfrak{R}}_p x^{p-1}$ on $(1, x)$ when $x \geq 1$ and $(x, 1)$ when $x < 1$ yields $(\underline{\mathfrak{R}}_p \mathbb{I}_{\{x \geq 1\}} + \overline{\mathfrak{R}}_p \mathbb{I}_{\{x<1\}})(x^p - 1/p) + U_p(1) \leq U_p(x) \leq (\overline{\mathfrak{R}}_p \mathbb{I}_{\{x \geq 1\}} + \underline{\mathfrak{R}}_p \mathbb{I}_{\{x<1\}})(x^p/p - 1/p) + U_p(1)$ for $x > 0$. Furthermore each U_p satisfies the Inada condition, i.e., $\lim_{x \downarrow 0} U_p'(x) = \infty$ and $\lim_{x \uparrow \infty} U_p'(x) = 0$, and U_δ has reasonable asymptotic elasticity, i.e., $AE_\infty(U_\delta) < 1$.

The discounted prices of risky assets are specified to be stochastic exponential $S = (\mathcal{E}(R^1), \cdots, \mathcal{E}(R^d))$, where R is an $\mathbb{R}^d$-valued càdlàg locally bounded semimartingale with $R_0 = 0$.

The agent is endowed with the utility random field $\mathcal{U}_p$ and an initial capital $x_0 \in \mathbb{R}_+$. A trading strategy is a predictable R-integrable $\mathbb{R}^d$-valued process π whose i-th component π^i represents the fraction of current wealth invested in the i-th risky asset. Then the associated wealth process $X(\pi)$ satisfies

$$X_t = x_0 + \int_0^t X_{s-} \pi_s \, dR_s, \quad 0 \leq t \leq T.$$

A trading strategy is *admissible* if the associated wealth process is strictly positive. We denote by $\mathcal{A}(x_0)$ the class of admissible trading strategies. For an admissible strategy π, $H^i := \pi^i X / S^i_- \mathbb{I}_{\{S^i_- \neq 0\}}$ corresponds to the number of shares invested in the i-th asset.

The agent chooses admissible trading strategies to maximize her utility of terminal wealth:

$$u_p(x_0) := \sup_{\pi \in \mathcal{A}(x_0)} \mathbb{E}_{\mathbb{P}} \left[D_T U_p(X_T(\pi)) \right]. \tag{1.3}$$

The dependence of u_p on x_0 will be omitted if no confusion is caused. Since U_p is bounded from above, $u_p(x_0) < \infty$ whenever D_T has finite $\mathbb{P}$-expectation. We recall the following version of Theorem 3.10 from Karatzas and Žitković (2003).

Proposition 1.17 (Karatzas-Žitković). *Assume that the set of equivalent local martingale measures for S is not empty, moreover there exist constants $0 < k_1 \leq k_2 < \infty$ such that $k_1 \leq D_t \leq k_2$, $0 \leq t \leq T$. Then for each $p < 0$ there exists an optimal strategy $\pi_p \in \mathcal{A}(x_0)$ for (1.3). The associated wealth process $X^{(p)}$ satisfies*

$$y_p Y_T^{(p)} = D_T U_p'(X_T^{(p)}),$$

where $y_p = u_p'(x_0)$ and $Y^{(p)}$ is some supermartingale deflator with $Y_0^{(p)} = 1$. Moreover

$$y_p\, x_0 = \mathbb{E}_{\mathbb{P}} \left[D_T U_p'(X_T^{(p)}) X_T^{(p)} \right] \geq \mathbb{E}_{\mathbb{P}} \left[D_T U_p'(X_T^{(p)}) X_T \right],$$

for any wealth process X associated to some admissible strategy.

To state our second main result, let us recall the exponential hedging problem. Given a contingent claim $B \in L^\infty(\mathcal{F}_T)$, the agent choose permissible strategy to maximize the expected exponential utility of the terminal wealth including the claim,

$$\sup_{\vartheta \text{ permissible}} \mathbb{E}_{\mathbb{P}} \left[-\exp(B - x_0 - (\vartheta \cdot R)_T) \right]. \tag{1.4}$$

Here ϑ is the monetary value invested in the risky assets. Its corresponding number of shares is $H^i := \vartheta^i / S^i_- \mathbb{I}_{\{S^i_- \neq 0\}}$ which satisfies $H \cdot S = \vartheta \cdot R$. The strategy ϑ is permissible if its corresponding $H \in \mathcal{H}^{perm}$. When S is locally bounded, (1.4) admits an optimal strategy $\hat{\vartheta}$; see (Kabanov and Stricker, 2002, Theorem 2.1).

To state the second main result, we impose the following assumption on the filtration. The most important example where this assumption holds in the Brownian filtration.

Assumption 1.18. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is continuous, i.e., all $\mathcal{F}$-local martingales are continuous.

Under the previous assumption, S is clearly continuous. Therefore R satisfies the *structure condition*:

$$R = M + \int d\langle M \rangle \lambda,$$

where M is a continuous local martingale with $M_0 = 0$ and $\lambda \in L^2_{loc}(M)$; see Schweizer (1995). The previous assumption also ensures the continuity of *opportunity processes* $L^{(p)}$ recalled in Section 3.

Our second main result studies the asymptotic behavior of the optimal strategy π_p for (1.1) as $p \downarrow -\infty$.

Theorem 1.19. *Let Assumptions 1.5, 1.15, and 1.18 hold. Set $D_T = \exp(B)$ for $B \in L^\infty(\mathcal{F}_T)$. If $\underline{\mathfrak{R}}_p$ and $\overline{\mathfrak{R}}_p$ in Assumption 1.15 satisfy*

$$\limsup_{p\downarrow -\infty} (1-p)\,(\overline{\mathfrak{R}}_p - 1) < \infty \qquad \textit{and} \qquad \limsup_{p\downarrow -\infty} (1-p)\,(1 - \underline{\mathfrak{R}}_p) < \infty, \tag{1.5}$$

then

$$\mathbb{P} - \lim_{p\downarrow -\infty} \int_0^T \left((1-p)\pi_p - \hat{\vartheta} \right)^\top_t d\langle M \rangle_t \left((1-p)\pi_p - \hat{\vartheta} \right)_t = 0.$$

This result states that whenever the ratio of marginal utilities converges to 1 at least as fast as the relative risk aversion converging to infinity, the optimal fraction invested in risky assets in the power type problem, after scaled by $1-p$, converges to the optimal monetary value invested in the exponential hedging problem. Here $(1-p)^{-1}$ can be considered as the rate of convergence.

Remark 1.20. Given a utility function U such that

$$\underline{\mathfrak{R}} \le \frac{U'(x)}{x^{p_0 - 1}} \le \overline{\mathfrak{R}}, \qquad \text{for all } x > 0,$$

where $0 < \underline{\mathfrak{R}} \le 1 \le \overline{\mathfrak{R}}$ and $p_0 < 0$, there exists a family of utilities $(U_p)_{p \le p_0}$ such that $U_{p_0} = U$ and (1.5) is satisfied for some sequences $(\overline{\mathfrak{R}}_p)_{p\le p_0}$ and $(\underline{\mathfrak{R}}_p)_{p \le p_0}$. Indeed, take any function $f : (-\infty, 0) \to (0,1)$ such that $f(p_0) = 1$ and $\limsup_{p\downarrow -\infty} (1-p)\, f(p) < \infty$. Set

$$U'_p(x) = f(p) x^{p-p_0} U'(x) + (1 - f(p)) x^{p-1}, \qquad \text{for } p \le p_0.$$

One can check that U_p is concave and continuous differentiable for each p moreover

$$\underline{\mathfrak{R}}_p := f(p)(\underline{\mathfrak{R}} - 1) + 1 \le \frac{U'_p(x)}{x^{p-1}} \le f(p)(\overline{\mathfrak{R}} - 1) + 1 =: \overline{\mathfrak{R}}_p,$$

where both $\limsup_{p\downarrow -\infty} (1-p)\,(1 - \underline{\mathfrak{R}}_p)$ and $\limsup_{p\downarrow-\infty} (1-p)\,(\overline{\mathfrak{R}}_p - 1)$ are finite.

Remark 1.21. Denote by $\widetilde{\pi}_p$ the optimal strategy for (1.3) when $U_p = x^p/p$. Nutz proved a remarkable result in (Nutz, 2012, Theorem 3.2) that $(1-p)\widetilde{\pi}_p \to \hat{\vartheta}$ in $L^2_{loc}(M)$; see (Nutz, 2012, Lemma A.3) for characterization of this convergence. In particular the previous convergence implies

$$\mathbb{P} - \lim_{p\downarrow -\infty} \int_0^T ((1-p)\widetilde{\pi}_p - \hat{\vartheta})_t^\top \, d\langle M\rangle_t \, ((1-p)\widetilde{\pi}_p - \hat{\vartheta})_t = 0. \tag{1.6}$$

Therefore $\widetilde{\pi}_p$ converges to $\hat{\vartheta}$ at the rate of $(1-p)^{-1}$. We complement Nutz's result by showing that $\pi_p - \widetilde{\pi}_p$ converges to 0 at the rate $(1-p)^{-1}$, when the ratio of marginal utilities converges to 1 at least at the same rate. In particular, we prove

$$\mathbb{P} - \lim_{p\downarrow -\infty} \int_0^T (1-p)(\widetilde{\pi}_p - \pi_p)_t^\top \, d\langle M\rangle_t \, (1-p)(\widetilde{\pi}_p - \pi_p)_t = 0. \tag{1.7}$$

Then Theorem 1.19 follows from combining the previous two convergence.

Remark 1.22. One can assume that both S and the opportunity process $L^{(p)}$ are continuous for all $p<0$ instead of Assumption 1.18 which is the most important and easy to check sufficient condition for the continuity of S and $(L^{(p)})_{p<0}$. Only the continuity of S is used to prove (1.7), continuity of both S and $L^{(p)}$ for all $p<0$ are needed for (1.6).

2. Stability for utilities defined on $\mathbb{R}$

We will prove Theorem 1.8 and its corollaries in this section. Let us start with the following property on the family $(\mathcal{M}^a_\delta)_{\delta\geq 0}$.

Lemma 2.1. *Under Assumption 1.2, all $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) are the same for all $\delta \geq 0$.*

Proof. Denote $\widetilde{U}_\delta(x) = -\frac{1}{\alpha_\delta}\exp(-\alpha_\delta x)$ and $\widetilde{V}_\delta(y) = \frac{1}{\alpha_\delta} y\log y - \frac{y}{\alpha_\delta}$ to be its convex conjugate. Here α_δ converges to $a_0 := 1$ as $\delta \downarrow 0$. Set $y = U'_\delta(x)$, which can take arbitrary value in $(0,\infty)$ as x varies in $\mathbb{R}$. It follows from Assumption 1.2 that $y/\overline{\mathfrak{R}} \leq \widetilde{U}'_\delta(-V'_\delta(y)) \leq y/\underline{\mathfrak{R}}$, which implies $\widetilde{V}'_\delta(y/\overline{\mathfrak{R}}) \leq V'_\delta(y) \leq \widetilde{V}'_\delta(y/\underline{\mathfrak{R}})$ for any $y \in (0,\infty)$. Integrating the previous inequality on $(0,y)$ and utilizing $\widetilde{V}_\delta(0) = \widetilde{U}_\delta(\infty) = 0$, we obtain

$$\overline{\mathfrak{R}}\,\widetilde{V}_\delta(y/\overline{\mathfrak{R}}) + V_\delta(0) \leq V_\delta(y) \leq \underline{\mathfrak{R}}\widetilde{V}_\delta(y/\underline{\mathfrak{R}}) + V_\delta(0).$$

Recall from Remark 1.3 that $(U_\delta(\infty))_{\delta>0}$ is uniformly bounded. Then there exists N such that $-N \leq V_\delta(0) = U_\delta(\infty) \leq N$ for any δ. The previous two inequalities combined yield

$$\frac{1}{\alpha_\delta}\widetilde{V}_0(y) - \frac{1}{\alpha_\delta} y\log\overline{\mathfrak{R}} - N \leq V_\delta(y) \leq \frac{1}{\alpha_\delta}\widetilde{V}_0(y) - \frac{1}{\alpha_\delta} y\log\underline{\mathfrak{R}} + N, \quad \text{for any } y.$$

Therefore $\mathbb{E}_{\mathbb{P}}[V_\delta(d\widetilde{\mathbb{P}}/d\mathbb{P})] < \infty$ if and only if $\mathbb{E}_{\mathbb{P}}[\widetilde{V}_0(d\widetilde{\mathbb{P}}/d\mathbb{P})] < \infty$. $\square$

To prove Theorem 1.8, observe that, without loss of generality all $(\alpha_\delta)_{\delta\geq 0}$ in Assumption 1.2 can be assumed to be 1. Indeed, define $\overline{U}_\delta(x) := \alpha_\delta U_\delta(x/\alpha_\delta)$. Assumption 1.2 implies

$$\underline{\mathfrak{R}} \leq \frac{\overline{U}'_\delta(x)}{\exp(-x)} \leq \overline{\mathfrak{R}}, \quad \text{for any } x\in\mathbb{R}.$$

Moreover, $\overline{U}(x)$ converges to $-\exp(-x)$ pointwise, since α_δ converges to 1 and $U_\delta(x)$ converges to $-\exp(-x)$ locally uniformly; see (Rockafellar, 1970, pp. 90). Therefore (1.1) can be rewritten as

$$u_\delta = \frac{1}{\alpha_\delta} \sup_{H \in \mathcal{H}^{perm}} \mathbb{E}_\mathbb{P}\left[\overline{U}_\delta\left((\alpha_\delta H \cdot S)_T + \alpha_\delta \xi_\delta\right)\right] = \frac{1}{\alpha_\delta} \sup_{\overline{H} \in \mathcal{H}^{perm}} \mathbb{E}_\mathbb{P}\left[\overline{U}_\delta\left((\overline{H} \cdot S)_T + \overline{\xi}_\delta\right)\right],$$

where $\overline{\xi}_\delta := \alpha_\delta \xi_\delta$. Therefore the optimal strategy H_δ for (1.1) is exactly $\overline{H}_\delta / \alpha_\delta$ where $\overline{H}_\delta$ maximizes the rightmost problem. Hence we can consider (1.1) with utility $\overline{U}_\delta$ and the random endowment $\overline{\xi}_\delta$. In this case Assumption 1.2 holds with $\alpha_\delta = 1$ for all $\delta \geq 0$.

Now suppose that Theorem 1.8 holds for $\overline{U}_\delta$, then the same statements hold for U_δ as well. For example, if $\lim_{\delta \downarrow 0} \mathbb{E}_\mathbb{Q}\left[\left|\left((\overline{H}_\delta - \overline{H}_0) \cdot S\right)_T\right|\right] = 0$, then

$$\begin{aligned} \mathbb{E}_\mathbb{Q}\left[|((H_\delta - H_0) \cdot S)_T|\right] &= \frac{1}{\alpha_\delta} \mathbb{E}_\mathbb{Q}\left[\left|\left((\overline{H}_\delta - \alpha H_0) \cdot S\right)_T\right|\right] \\ &\leq \frac{1}{\alpha_\delta} \mathbb{E}_\mathbb{Q}\left[\left|\left((\overline{H}_\delta - \overline{H}_0) \cdot S\right)_T\right|\right] + \frac{|\alpha_\delta - 1|}{\alpha_\delta} \mathbb{E}_\mathbb{Q}\left[|(H_0 \cdot S)_T|\right] \\ &\to 0, \quad \text{as } \delta \downarrow 0, \end{aligned} \tag{2.1}$$

where $\overline{H}_0 = H_0$ is used in the inequality and $\mathbb{E}_\mathbb{Q}[|(H_0 \cdot S)_T|] < \infty$ since $H_0 \cdot S$ is a $\mathbb{Q}$-martingale. Therefore, thanks to the previous change of variable, it suffices to prove Theorem 1.8 when

$$\alpha_\delta = 1, \quad \text{for all } \delta \geq 0.$$

To this end, Theorem 1.8 i) will be proved in Corollary 2.3, ii) in Proposition 2.7, and iii) in Corollary 2.5.

In the rest of this section, Assumptions 1.1-1.5 and 1.7 are enforced. To simplify notation, we introduce

$$X^\delta := H_\delta \cdot S, \quad \mathcal{X}^\delta := X^\delta + \xi_\delta, \quad \Delta\xi_\delta := \xi_\delta - \xi_0, \quad \text{and} \quad \Delta\mathcal{X}^\delta := \mathcal{X}^\delta - \mathcal{X}^0, \quad \text{for } \delta \geq 0.$$

Proof of Theorem 1.8 i) starts with the following estimate.

Lemma 2.2. *It holds that*

$$\lim_{\delta \downarrow 0} \mathbb{E}_\mathbb{Q}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}_T^\delta) \exp(-\Delta\mathcal{X}_T^\delta)\right| \left|\Delta X_T^\delta\right|\right] = 0$$

Proof. Recall from Proposition 1.6 that X^0 is a $\mathbb{Q}_\delta$-supermartingale and X^δ is a $\mathbb{Q}_\delta$-martingale, where the density of $\mathbb{Q}_\delta \in \mathcal{M}^e$ with respect to $\mathbb{P}$ is $U'_\delta(\mathcal{X}_T^\delta)$ up to a constant. Therefore $U'_\delta(\mathcal{X}_T^\delta) X^0_\cdot$ is a $\mathbb{P}$-supermartingale and $U'_\delta(\mathcal{X}_T^\delta) X^\delta_\cdot$ is a $\mathbb{P}$-martingale. Since both these two processes have initial value zero, therefore $\mathbb{E}_\mathbb{P}\left[U'_\delta(\mathcal{X}_T^\delta) X_T^0\right] \leq 0 = \mathbb{E}_\mathbb{P}\left[U'_\delta(\mathcal{X}_T^\delta) X_T^\delta\right]$, which induces

$$\mathbb{E}_\mathbb{P}\left[U'_\delta(\mathcal{X}_T^\delta)(X_T^0 - X_T^\delta)\right] \leq 0.$$

Similarly, the previous argument applied to $\mathbb{Q}$ gives

$$\mathbb{E}_\mathbb{P}\left[U'_0(\mathcal{X}_T^0)(X_T^\delta - X_T^0)\right] \leq 0.$$

Summing up the previous two inequalities and changing to the measure $\mathbb{Q}$ whose density is $U_0'(\mathcal{X}_T^0)$ up to a constant, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\left(1-\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\right)\left(X_T^\delta - X_T^0\right)\right] \leq 0.$$

Observe that the random variable in the expectation of the previous inequality is negative only when $X_T^0 \geq X_T^\delta \geq I_\delta\left(U_0'(\mathcal{X}_T^0)\right) - \xi_\delta$ or $I_\delta\left(U_0'(\mathcal{X}_T^0)\right) - \xi_\delta \geq X_T^\delta \geq X_T^0$, where $I_\delta = (U_\delta')^{-1}$. In either cases,

$$\left(\left(1-\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\right)\left(X_T^\delta - X_T^0\right)\right)_- \leq \left(\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)} - 1\right)\left(I_\delta\left(U_0'(\mathcal{X}_T^0)\right) - \xi_\delta - X_T^0\right),$$

where $(\cdot)_-$ represents the negative part. Utilizing the fact that $\mathbb{E}_{\mathbb{Q}}[|A|] \leq 2\mathbb{E}_{\mathbb{Q}}[A_-]$ for any random variable A with $\mathbb{E}_{\mathbb{Q}}[A] \leq 0$, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\left|\left(1-\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\right)\left(X_T^\delta - X_T^0\right)\right|\right] \leq 2\,\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)} - 1\right)\left(I_\delta\left(U_0'(\mathcal{X}_T^0)\right) - \xi_\delta - X_T^0\right)\right].$$

Note that the left side of the previous inequality is $\mathbb{E}_{\mathbb{Q}}\left[\left|1-\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right| |\Delta X_T^\delta|\right]$. The statement follows once the expectation on the right side converges to zero as $\delta \downarrow 0$.

To prove the desired convergence, let us first estimate the upper bound of $|I_\delta(U_0'(x)) - x|$ on $\mathbb{R}$. Set $y = U_0'(x)$. It follows

$$\begin{aligned} I_\delta\left(U_0'(x)\right) - x &= I_\delta(y) - I_0(y) = -\log\left[\exp\left(-\left(I_\delta(y) - I_0(y)\right)\right)\right] \\ &= -\log\left[\frac{\exp(-I_\delta(y))}{y}\right] = \log\left[\frac{U_\delta'(I_\delta(y))}{U_0'(I_\delta(y))}\right]. \end{aligned}$$

Assumption 1.2 then implies

$$\sup_{x\in\mathbb{R}}\left|I_\delta(U_0'(x)) - x\right| \leq \max\{\log\overline{\mathfrak{R}}, \log 1/\underline{\mathfrak{R}}\} =: \eta.$$

As a result, $\left|I_\delta(U_0'(\mathcal{X}_T^0)) - X_T^0 - \xi_\delta\right| \leq \eta + |\Delta\xi_\delta|$. Assumptions 1.2 and 1.7 combined imply that

$$\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)} = \mathfrak{R}_\delta(X_T^0+\xi_\delta)\exp(-\Delta\xi_\delta) \leq \overline{\mathfrak{R}}e^C.$$

The previous two estimates combined yield

$$\left|\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)} - 1\right|\left|I_\delta\left(U_0'(\mathcal{X}_T^0) - \xi_\delta - X_T^0\right)\right| \leq (\overline{\mathfrak{R}}e^C + 1)(\eta + |\Delta\xi_\delta|),$$

where the right side is uniformly integrable in δ under $\mathbb{Q}$ thanks to $\mathbb{E}_{\mathbb{Q}}[|\Delta\xi_\delta|] = 0$ in Assumption 1.7. On the other hand, the term on the left side of the previous inequality converges to 0 in probability $\mathbb{Q}$. This follows from facts that $\lim_{\delta\downarrow 0}|I_\delta(U_0'(\mathcal{X}_T^0)) - \xi_\delta - X_T^0|$ is bounded and $\mathbb{Q}-\lim_{\delta\downarrow 0}\mathfrak{R}_\delta(X_T^0 + \xi_\delta)\exp(-\Delta\xi_\delta) = 1$. The previous convergence follows from

$$\begin{aligned} &\mathbb{Q}\left(|\mathfrak{R}_\delta(X_T^0+\xi_\delta)\exp(-\Delta\xi_\delta) - 1| \geq \epsilon\right) \\ \leq &\mathbb{Q}\left(|\mathfrak{R}_\delta(X_T^0+\xi_\delta)\exp(-\Delta\xi_\delta) - 1| \geq \epsilon, |\xi_\delta| \leq N, |X_T^0| \leq N\right) + \mathbb{Q}(|\xi_\delta| > N) + \mathbb{Q}(|X_T^0| > N), \end{aligned}$$

where the first term on the right converges to 0 as $\delta \downarrow 0$ since $\mathfrak{R}_\delta$ converges to 1 locally uniformly and $\mathbb{Q} - lim_{\delta\downarrow 0}\Delta\xi_\delta = 0$, both second and third terms can be made arbitrarily small for sufficiently large N. The uniform integrability and convergence in probability combined imply that

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left|\frac{U'_\delta(X^0_T+\xi_\delta)}{U'_0(X^0_T+\xi_0)}-1\right|\left|I_\delta\left(U'_0(\mathcal{X}^0_T)-\xi_\delta-X^0_T\right)\right|\right]=0,$$

hence the statement. □

The previous result provides a handle to study the $\mathbb{L}^1(\mathbb{Q})$ convergence of $X^\delta_T - X^0_T$.

Corollary 2.3. *It holds that*

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left|\Delta X^\delta_T\right|\right]=0.$$

Proof. We will first prove

$$\lim_{\delta\downarrow 0}\mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T|\geq\epsilon, |\mathcal{X}^\delta_T|\leq N\right)=0, \qquad \text{for any } \epsilon, N>0. \tag{2.2}$$

To this end, for fixed ϵ and N, $\exp(-\Delta\mathcal{X}^\delta_T)\leq e^{-\epsilon}$ when $\Delta\mathcal{X}^\delta_T\geq\epsilon$. Since U'_δ converges to U'_0 locally uniformly, there exists a sufficiently small δ such that $e^{-\epsilon/2}\leq\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\leq e^{\epsilon/2}$ for $|\mathcal{X}^\delta_T|\leq N$. On the other hand, $|\Delta X^\delta_T|\geq\epsilon/2$ when $|\Delta\xi_\delta|\leq\epsilon/2$ and $|\Delta\mathcal{X}^\delta_T|\geq\epsilon$. The previous estimates combined imply that on $\{\Delta\mathcal{X}^\delta_T\geq\epsilon, |\Delta\xi_\delta|\leq\epsilon/2, |\mathcal{X}^\delta_T|\leq N\}$,

$$\left|1-\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right||\Delta X^\delta_T|\geq(1-e^{\epsilon/2}e^{-\epsilon})\epsilon/2>0, \qquad \text{for sufficiently small } \delta.$$

Similarly, on $\{\Delta\mathcal{X}^\delta_T\leq-\epsilon, |\Delta\xi_\delta|\leq\epsilon/2, |\mathcal{X}^\delta_T|\leq N\}$,

$$\left|1-\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right||\Delta X^\delta_T|\geq(e^{-\epsilon/2}e^{\epsilon}-1)\epsilon/2>0, \qquad \text{for sufficiently small } \delta.$$

Set $\eta=\min\{1-e^{-\epsilon/2}, e^{\epsilon/2}-1\}\cdot\epsilon/2>0$. The previous two inequalities combined yield

$$\eta\cdot\mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T|\geq\epsilon, |\Delta\xi_\delta|\leq\epsilon/2, |\mathcal{X}^\delta_T|\leq N\right)\leq\mathbb{E}_{\mathbb{Q}}\left[\left|1-\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right||\Delta X^\delta_T|\right]\to 0, \qquad \text{as } \delta\downarrow 0,$$

where the convergence follows from Lemma 2.2. Therefore (2.2) follows from the previous inequality and $\lim_{\delta\downarrow 0}\mathbb{Q}(|\Delta\xi_\delta|>\epsilon/2)=0$.

Second, we will prove

$$\lim_{\delta\downarrow 0}\mathbb{Q}(|\Delta\mathcal{X}^\delta_T|\geq\epsilon)=0. \tag{2.3}$$

To this end, note that

$$\begin{aligned}\mathbb{Q}(|\mathcal{X}^\delta_T|\geq N)&\leq\mathbb{Q}(|\mathcal{X}^\delta_T|\geq N, |\mathcal{X}^0_T|\leq N/2)+\mathbb{Q}(|\mathcal{X}^0_T|\geq N/2)\\&\leq\mathbb{Q}(|\Delta\mathcal{X}^\delta_T|\geq N/2)+\mathbb{Q}(|\mathcal{X}^0_T|\geq N/2), \qquad \text{for any } N.\end{aligned}\tag{2.4}$$

Let us prove in what follows

$$\lim_{\delta\downarrow 0}\mathbb{Q}(|\Delta\mathcal{X}^\delta_T|\geq N/2)=0, \qquad \text{for sufficiently large } N. \tag{2.5}$$

Take $N/2>\max\{2,\log 1/\underline{\mathfrak{R}},\log\overline{\mathfrak{R}}\}$ and set $M^\delta=N/2\vee(|\Delta\xi_\delta|+1)$. On $\{\Delta\mathcal{X}^\delta_T\leq-M^\delta\}$, $\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\geq\underline{\mathfrak{R}}\exp(N/2)>1$. Hence on the same set,

$$\left|1-\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right||\Delta X^\delta_T|\geq\left(\underline{\mathfrak{R}}\exp\left(N/2\right)-1\right)|\Delta\mathcal{X}^\delta_T-\Delta\xi_\delta|\geq\underline{\mathfrak{R}}\exp\left(N/2\right)-1.$$

On $\{\Delta\mathcal{X}^\delta_T \geq M^\delta\}$, $\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T) \leq \overline{\mathfrak{R}}\exp(-N/2) < 1$. Hence on the same set,

$$\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right| |\Delta X^\delta_T| \geq 1 - \overline{\mathfrak{R}}\exp\left(-N/2\right).$$

Set $\eta = \min\{\underline{\mathfrak{R}}\exp(N/2) - 1, 1 - \overline{\mathfrak{R}}\exp(-N/2)\} > 0$. The previous two inequalities combined yield

$$\begin{aligned}\eta \cdot \mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq M^\delta\right) &\leq \eta\,\mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\,\mathbb{I}_{\{|\Delta\mathcal{X}^\delta_T| \geq M^\delta\}}\right]\\ &\leq \mathbb{E}_\mathbb{Q}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\right| |\Delta X^\delta_T|\,\mathbb{I}_{\{\Delta\mathcal{X}^\delta_T \geq M^\delta\}}\right]\\ &\to 0, \quad \text{as } \delta \downarrow 0,\end{aligned} \tag{2.6}$$

where the convergence follows from Lemma 2.2. Therefore (2.5) follows from

$$\begin{aligned}\mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq N/2\right) &\leq \mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq N/2, |\Delta\xi_\delta| \leq 1\right) + \mathbb{Q}\left(|\Delta\xi_\delta| > 1\right)\\ &= \mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq M^\delta, |\Delta\xi_\delta| \leq 1\right) + \mathbb{Q}\left(|\Delta\xi_\delta| > 1\right)\\ &\to 0, \quad \text{as } \delta \downarrow 0.\end{aligned}$$

Switch our attention to $\mathbb{Q}(|\mathcal{X}^0_T| \geq N/2)$. Assumption 1.4 yields $x_0 \leq \mathbb{E}_\mathbb{Q}[\xi_0] \leq \widetilde{x}_0 + \mathbb{E}_\mathbb{Q}[(G_0 \cdot S)_T] \leq \widetilde{x}_0$, where $G_0 \cdot S$ is a $\mathbb{Q}$-local martingale bounded from below hence a $\mathbb{Q}$-supermartingale. Moreover recall that X^0 is a $\mathbb{Q}$-martingale. Therefore $\mathbb{Q}(|\mathcal{X}^0_T| \geq N/2) \leq 2\,\mathbb{E}_\mathbb{Q}[|\mathcal{X}^0_T|]/N$ which can be made arbitrarily small for sufficiently large N. The previous inequality combined with (2.4) and (2.5) yields that $\limsup_{\delta\downarrow 0}\mathbb{Q}(|\mathcal{X}^\delta_T| \geq N)$ is sufficiently small for large N. Hence (2.3) follows from combining the previous limit superior with (2.2).

Finally, we will prove

$$\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\left|\Delta X^\delta_T\right|\right] = 0.$$

To this end, we have seen in (2.6) that $\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\left|\Delta X^\delta_T\right|\,\mathbb{I}_{\{|\Delta\mathcal{X}^\delta_T| \geq M^\delta\}}\right] = 0$. On the other hand, $\mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\,\mathbb{I}_{\{|\Delta X^\delta_T| < M^\delta\}}\right] \leq \mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\,\mathbb{I}_{\{|\Delta X^\delta_T| < M^\delta, |\Delta\xi_\delta| \leq 1\}}\right] + \mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\,\mathbb{I}_{\{|\Delta X^\delta_T| < M^\delta, |\Delta\xi_\delta| > 1\}}\right]$. Here the second term on the right is bounded from above by $\frac{N}{2}\mathbb{Q}(|\Delta\xi_\delta| > 1) + \mathbb{E}_\mathbb{Q}\left[(|\Delta\xi_\delta| + 1)\,\mathbb{I}_{\{|\Delta\xi_\delta| > 1\}}\right]$ which converges to 0 as $\delta \downarrow 0$ thanks to Assumption 1.7. The first term converges to 0 as well. Indeed, since $|\Delta X^\delta_T| \leq N/2 + 1$ when $|\Delta\mathcal{X}^\delta_T| < M^\delta$ and $|\Delta\xi_\delta| \leq 1$, the bounded convergence theorem implies that $\lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\,\mathbb{I}_{\{|\Delta X^\delta_T| < M^\delta, |\Delta\xi_\delta| \leq 1\}}\right] = 0$ along any subsequence of δ such that ΔX^δ_T converges to 0 $\mathbb{Q}$-a.s.. Recall $\mathbb{Q} - \lim_{\delta\downarrow 0}\Delta X^\delta_T = 0$ from (2.3). The previous convergence holds along the entire sequence of δ. This argument which combines convergence in probability with the bounded convergence theorem will be used frequently later without mentioned explicitly. □

Now we are going to prove Theorem 1.8 iii) via the $\mathbb{L}^1(\mathbb{Q})$-convergence of ΔX^δ_T. Let us first prepare the following result.

Lemma 2.4. *For any $\mathbb{Q}$-supermartingale Z with $Z_0 = 0$,*

$$\mathbb{E}_\mathbb{Q}\left[\sup_{0\leq t\leq T}|Z_t|^p\right] \leq \frac{1}{1-p}2^p\,\mathbb{E}_\mathbb{Q}\left[|Z_T|\right]^p, \qquad \textit{for any } p \in (0,1).$$

Proof. Doob's first submartingale inequality (see (Karatzas and Shreve, 1991, Theorem 1.3.8)) implies that

$$\lambda\,\mathbb{Q}\left(\inf_{0\le t\le T} Z_t \le -\lambda\right) = \lambda\,\mathbb{Q}\left(\sup_{0\le t\le T}(-Z_t) \ge \lambda\right) \le \mathbb{E}_{\mathbb{Q}}\left[(-Z_T)_+\right] \le \mathbb{E}_{\mathbb{Q}}\left[|Z_T|\right], \quad \text{for any } \lambda > 0.$$

Meanwhile Doob's second submartingale inequality yields

$$\lambda\,\mathbb{Q}\left(\sup_{0\le t\le T} Z_t \ge \lambda\right) = \lambda\,\mathbb{Q}\left(\inf_{0\le t\le T}(-Z_t) \le -\lambda\right) \le \mathbb{E}_{\mathbb{Q}}[(-Z_T)_+] - \mathbb{E}_{\mathbb{Q}}[-Z_0] \le \mathbb{E}_{\mathbb{Q}}[|Z_T|], \quad \text{for any } \lambda > 0.$$

Since $\{\sup_{0\le t\le T}|Z_t| \ge \lambda\} = \{\sup_{0\le t\le T} Z_t \ge \lambda\} \cup \{\inf_{0\le t\le T} Z_t \le -\lambda\}$, the previous two inequalities combined imply

$$\lambda\,\mathbb{Q}\left(\sup_{0\le t\le T}|Z_t| \ge \lambda\right) \le 2\,\mathbb{E}_{\mathbb{Q}}[|Z_T|].$$

Set $Z_* = \sup_{0\le t\le T}|Z_t|$. It then follows

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\le t\le T}|Z_t|^p\right] &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \mathbb{I}_{\{Z_*>x\}} p x^{p-1}\,dx\right] = \int_0^\infty \mathbb{Q}(Z_* > x) p x^{p-1}\,dx \\ &\le \int_0^\infty \min\left\{1, \frac{2\,\mathbb{E}_{\mathbb{Q}}[|Z_T|]}{x}\right\} p x^{p-1}\,dx = \frac{1}{1-p} 2^p\,\mathbb{E}_{\mathbb{Q}}[|Z_T|]^p.\end{aligned}$$

□

Applying the previous lemma to the $\mathbb{Q}$-supermartingale ΔX^δ, we obtain

Corollary 2.5. *If S is continuous, then*

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left[\Delta X^\delta, \Delta X^\delta\right]_T^{p/2}\right] = 0, \qquad \textit{for any } p \in (0,1).$$

Proof. Corollary 2.3 and Lemma 2.4 combined imply that $\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\le t\le T}|\Delta X_t^\delta|^p\right] = 0$. If S is continuous, ΔX^δ is a continuous $\mathbb{Q}$-local martingale. The statement then follows from Burkholder-Davis-Gundy inequality; see (Rogers and Williams, 1987, Chapter IV, Theorem 42.1). □

The following result prepares the proof of Theorem 1.8 ii).

Lemma 2.6. *It holds that*

$$\lim_{\delta\downarrow 0}\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\mathcal{X}_T^\delta\right)\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\mathcal{X}_T^0\right)\right]} = 1.$$

Proof. Proposition 1.6 implies that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp(-\mathcal{X}_T^0)}{\mathbb{E}_{\mathbb{P}}[\exp(-\mathcal{X}_T^0)]}.$$

After changing to the measure $\mathbb{Q}$, the statement is equivalent to

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}[\exp(-\Delta\mathcal{X}_T^\delta)] = 1. \tag{2.7}$$

Fix $N > \max\{C, \log 1/\underline{\mathfrak{R}}\}$ where C is the constant in Assumption 1.7. We have seen in Lemma 2.3 that $\mathbb{Q} - \lim_{\delta\downarrow 0} \Delta\mathcal{X}_T^\delta = 0$. The bounded convergence theorem then yields

$$\lim_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta \geq -N\}}\right] = 1. \tag{2.8}$$

On the other hand, when $\Delta\mathcal{X}_T^\delta \leq -N$, $\Delta X_T^\delta = \Delta\mathcal{X}_T^\delta - \Delta\xi_\delta \leq -N + C < 0$, then

$$\begin{aligned}\left|1 - \mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right| |\Delta X_T^\delta| &= \exp(-\Delta\mathcal{X}_T^\delta)\left|\exp(\Delta\mathcal{X}_T^\delta) - \mathfrak{R}_\delta(\mathcal{X}_T^\delta)\right| |\Delta X_T^\delta| \\ &\geq \exp(-\Delta\mathcal{X}_T^\delta)\,(\underline{\mathfrak{R}} - \exp(-N))\,(N - C).\end{aligned}$$

Set $\eta = (\underline{\mathfrak{R}} - \exp(-N))\,(N-C) > 0$. It then follows

$$\eta\cdot\mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta \leq -N\}}\right] \leq \mathbb{E}_\mathbb{Q}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right| |\Delta X_T^\delta|\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta \leq -N\}}\right] \to 0, \quad \text{as } \delta\downarrow 0, \tag{2.9}$$

where the convergence follows from Lemma 2.2. As a result, (2.7) follows from combining (2.8) and (2.9). □

Now we are ready to prove Theorem 1.8 ii).

Proposition 2.7. *It holds that*

$$\lim_{\delta\downarrow 0} u_\delta = u_0.$$

Proof. After changing to the measure $\mathbb{Q}$, the statement is equivalent to

$$1 = \lim_{\delta\downarrow 0}\frac{\mathbb{E}_\mathbb{P}\left[U_\delta(\mathcal{X}_T^\delta)\right]}{\mathbb{E}_\mathbb{P}\left[U_0(\mathcal{X}_T^0)\right]} = \lim_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right].$$

In what follows, we will prove

$$\limsup_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right] \leq 1. \tag{2.10}$$

While $\liminf_{\delta\downarrow 0}\mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right] \geq 1$ can be proved similarly. To prove (2.10), we will estimate the limit superior of the expectation on sets $\{-N \leq \mathcal{X}_T^\delta \leq N\}$, $\{\mathcal{X}_T^\delta > N\}$, and $\{\mathcal{X}_T^\delta < -N\}$ separately, for a fixed $N > 0$, in the following three steps.

Step 1: For any $\epsilon, N > 0$, there exists $\delta_{\epsilon,N}$ such that $1 - \epsilon \leq \frac{U_\delta'(x)}{U_0'(x)} \leq 1 + \epsilon$ for $x \in (-N, N)$ and $\delta \leq \delta_{\epsilon,N}$. Integrating $U_\delta'(x) \leq (1+\epsilon)U_0'(x)$ on (x, N) gives $U_\delta(x) \geq (1+\epsilon)U_0(x) - (1+\epsilon)U_0(N) + U_\delta(N)$, which yields

$$\frac{U_\delta(x)}{U_0(x)} \leq 1 + \epsilon + \frac{U_\delta(N) - (1+\epsilon)U_0(N)}{U_0(x)}, \quad \text{for } x \in [-N, N] \text{ and } \delta \leq \delta_{\epsilon,N}.$$

It then follows that

$$\begin{aligned}&\mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\,\mathbb{I}_{\{-N\leq\mathcal{X}_T^\delta\leq N\}}\right] = \mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^\delta)}\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\leq\mathcal{X}_T^\delta\leq N\}}\right] \\ &\leq (1+\epsilon)\,\mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\leq\mathcal{X}_T^\delta\leq N\}}\right] + (U_\delta(N) - (1+\epsilon)U_0(N))\,\mathbb{E}_\mathbb{Q}\left[\frac{\mathbb{I}_{\{-N\leq\mathcal{X}_T^\delta\leq N\}}}{U_0(\mathcal{X}_T^0)}\right] \\ &= (1+\epsilon)\,\mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\leq\mathcal{X}_T^\delta\leq N\}}\right] + (U_\delta(N) - (1+\epsilon)U_0(N))\,\frac{\mathbb{P}(-N \leq \mathcal{X}_T^\delta \leq N)}{\mathbb{E}_\mathbb{P}\left[U_0(\mathcal{X}_T^0)\right]}.\end{aligned}$$

In what follows the two terms on the right side of the previous inequality will be estimated separately.

First, note that $\Delta\mathcal{X}^\delta_T \geq -3N$ when $-N \leq \mathcal{X}^\delta_T \leq N$ and $\mathcal{X}^0_T \leq 2N$. Then the dominated convergence theorem and (2.3) combined yield that

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{-N\leq \mathcal{X}^\delta_T\leq N, \mathcal{X}^0_T\leq 2N\}}\right] = \mathbb{Q}(-N \leq \mathcal{X}^0_T \leq N).$$

Second, $\Delta\mathcal{X}^\delta_T \leq -N$ when $-N \leq \mathcal{X}^\delta_T \leq N$ and $\mathcal{X}^0_T > 2N$. Then

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{-N\leq \mathcal{X}^\delta_T\leq N, \mathcal{X}^0_T> 2N\}}\right] \leq \lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{\Delta\mathcal{X}^\delta_T\leq -N\}}\right] \to 0, \quad \text{as } \delta\downarrow 0,$$

where the last convergence holds owing to (2.9). The previous two convergence combined imply

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{-N\leq \mathcal{X}^\delta_T\leq N\}}\right] = \mathbb{Q}(-N \leq \mathcal{X}^0_T \leq N). \tag{2.11}$$

Third, $\lim_{\delta\downarrow 0} U_\delta(N) = U_0(N)$, (2.3) and $\mathbb{Q}\sim\mathbb{P}$ combined imply that

$$\lim_{\delta\downarrow 0} \left(U_\delta(N) - (1+\epsilon)U_0(N)\right) \frac{\mathbb{P}(-N\leq \mathcal{X}^\delta_T\leq N)}{\mathbb{E}_{\mathbb{P}}[U_0(\mathcal{X}^0_T)]} = -\epsilon\, U_0(N)\frac{\mathbb{P}(-N\leq \mathcal{X}^0_T\leq N)}{\mathbb{E}_{\mathbb{P}}[U_0(\mathcal{X}^0_T)]}. \tag{2.12}$$

<u>*Step 2:*</u> Integrating $\underline{\mathfrak{R}}U_0'(x) \leq U_\delta'(x)$ on (N,x) yields that $\underline{\mathfrak{R}}U_0(x) - \underline{\mathfrak{R}}U_0(N) + U_\delta(N) \leq U_\delta(x)$ for $x\geq N$. This implies that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}^\delta_T)}{U_0(\mathcal{X}^0_T)}\,\mathbb{I}_{\{\mathcal{X}^\delta_T\geq N\}}\right] \leq \underline{\mathfrak{R}}\,\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{\mathcal{X}^\delta_T\geq N\}}\right] + \left(U_\delta(N) - \underline{\mathfrak{R}}U_0(N)\right)\frac{\mathbb{P}(\mathcal{X}^\delta_T\geq N)}{\mathbb{E}_{\mathbb{P}}[U_0(\mathcal{X}^0_T)]}.$$

Lemma 2.6 and (2.11) combined give

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{\mathcal{X}^\delta_T\geq N, \mathcal{X}^\delta_T\leq -N\}}\right] = \mathbb{Q}(\mathcal{X}^0_T \geq N, \mathcal{X}^0_T\leq -N). \tag{2.13}$$

On the other hand, $\overline{\mathfrak{R}}U_0(N) \leq U_\delta(N) \leq \underline{\mathfrak{R}}U_0(N) < 0$ for sufficiently small δ. Combining the previous inequality with $U_\delta(N) - \underline{\mathfrak{R}}U_0(N) \geq U_\delta(0) - \underline{\mathfrak{R}}U_0(0)$, we obtain $0 \geq U_\delta(N) - \underline{\mathfrak{R}}U_0(N) \geq U_\delta(0) - \underline{\mathfrak{R}}U_0(0)$, where the right side is bounded uniformly in δ. Utilizing (2.3) and $\mathbb{Q}\sim\mathbb{P}$, we obtain

$$\limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}^\delta_T)}{U_0(\mathcal{X}^0_T)}\,\mathbb{I}_{\{\mathcal{X}^\delta_T\geq N\}}\right] \leq \underline{\mathfrak{R}}\,\mathbb{Q}(\mathcal{X}^0_T\geq N, \mathcal{X}^0_T\leq -N) + (1-\underline{\mathfrak{R}})\,U_0(0)\frac{\mathbb{P}(\mathcal{X}^0_T\geq N)}{\mathbb{E}_{\mathbb{P}}[U_0(\mathcal{X}^0_T)]}. \tag{2.14}$$

<u>*Step 3:*</u> Integrating $U_\delta'(x) \leq \overline{\mathfrak{R}}U_0'(x)$ on $(x,-N)$ gives $U_\delta(x) \geq \overline{\mathfrak{R}}U_0(x) + U_\delta(-N) - \overline{\mathfrak{R}}U_0(-N) \geq \overline{\mathfrak{R}}U_0(x)$, where the second inequality holds since $U_\delta(-N) \geq \overline{\mathfrak{R}}U_0(-N)$ for sufficiently small δ. As a result,

$$\limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}^\delta_T)}{U_0(\mathcal{X}^0_T)}\,\mathbb{I}_{\{\mathcal{X}^\delta_T\leq -N\}}\right] \leq \overline{\mathfrak{R}}\limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{\mathcal{X}^\delta_T\leq -N\}}\right] \leq \overline{\mathfrak{R}}\,\mathbb{Q}(\mathcal{X}^0_T\geq N, \mathcal{X}^0_T\leq -N). \tag{2.15}$$

Finally combining (2.11) — (2.15), (2.10) follows after sending $\epsilon\downarrow 0$ then $N\uparrow\infty$. $\square$

Proof of Corollary 1.10. Following the discussion after Lemma 2.1, we consider problem (1.1) for $\overline{U}_\delta(\alpha_\delta x)$ and $\overline{\xi}_\delta = \alpha_\delta x_0$. After the previous change of variable $f(\delta) = \sup_{x\in\mathbb{R}} |\overline{\mathfrak{R}}_\delta(x) - 1|$ where

$\overline{\mathfrak{R}}_\delta(x) = \overline{U}'_\delta(x)/\exp(-x)$. In what follows, we add a bar to random variables and processes associated to the problem for $\overline{U}_\delta$. In the rest of the proof, C represents a constant which may vary from line to line.

First, we utilize the argument in Lemma 2.2 to prove

$$\limsup_{\delta\downarrow 0}\,(f(\delta)+g(\delta))^{-2}\cdot\mathbb{E}_{\mathbb{Q}}\left[\left|1-\overline{\mathfrak{R}}_\delta(\overline{\mathcal{X}}^\delta_T)\exp(-\Delta\overline{\mathcal{X}}^\delta_T)\right|\,|\Delta\overline{X}^\delta_T|\right]<\infty.$$

To this end, we have seen in Lemma 2.2 that the expectation on the left side is bounded from above by

$$2\,\mathbb{E}_{\mathbb{Q}}\left[\left(\overline{\mathfrak{R}}_\delta(\overline{X}^0_T+\alpha_\delta x_0)\exp(-(\alpha_\delta-1)x_0)-1\right)\left(\overline{I}_\delta\left(\overline{U}'_0(\overline{\mathcal{X}}^0_T)\right)-\overline{\mathcal{X}}^0_T-(\alpha_\delta-1)x_0\right)\right]. \tag{2.16}$$

In expectation above, for sufficiently small δ, $1-f(\delta)\leq\overline{\mathfrak{R}}_\delta\leq 1+f(\delta)$ and $1-2g(\delta)x_0\leq\exp(-g(\delta)x_0)\leq\exp(-(\alpha_\delta-1)x_0)\leq\exp(g(\delta)x_0)\leq 1+2g(\delta)x_0$, where second inequality uses the facts that $e^y=1+\int_0^y e^z\,dz\leq 1+2y$ for $0<y\leq\ln 2$ and $e^{-y}\geq 1-y$ for $y>0$. Therefore

$$\left|\overline{\mathfrak{R}}_\delta(\overline{X}^0_T+\alpha_\delta x_0)\exp(-(\alpha_\delta-1)x_0)-1\right|\leq f(\delta)+2g(\delta)x_0+2f(\delta)g(\delta)x_0\leq C\left(f(\delta)+g(\delta)\right),\quad \mathbb{Q}\text{-}a.s., \tag{2.17}$$

for sufficiently small δ. On the other hand, we have seen in Lemma 2.2 that $\overline{I}_\delta(\overline{U}'_0(\overline{x}))-\overline{x}=\log\overline{\mathfrak{R}}_\delta(\overline{I}_\delta(\overline{y}))$ where $\overline{y}=\overline{U}'_0(\overline{x})$. It then follows $-2f(\delta)\leq\overline{I}_\delta(\overline{U}'_0(\overline{x}))-\overline{x}\leq 2f(\delta)$, where we use $\log(1-y)=-\int_{-y}^0(1+z)^{-1}dz\geq -2y$ for $0<y<1/2$ and $\log(1+y)\leq y$ for $y>0$. As a result

$$\left|\overline{I}_\delta\left(\overline{U}'_0(\overline{\mathcal{X}}^0_T)\right)-\overline{\mathcal{X}}^0_T-(\alpha_\delta-1)x_0\right|\leq 2f(\delta)+g(\delta)x_0\leq C(f(\delta)+g(\delta)),\quad \mathbb{Q}-a.s., \tag{2.18}$$

for sufficiently small δ. Combining (2.17) and (2.18), we obtain that the expectation in (2.16) is bounded from above by $C(f(\delta)+g(\delta))^2$ for sufficiently small δ. This confirms the claim.

In the next step, we will prove

$$\limsup_{\delta\downarrow 0}\,(f(\delta)+g(\delta))^{-2}\cdot\mathbb{E}_{\mathbb{Q}}\left[|\Delta\overline{X}^\delta_T|\right]<\infty. \tag{2.19}$$

Indeed, an argument similar to that in Corollary 2.3 implies that there exists $N,\eta>0$ such that

$$\eta\,\mathbb{E}_{\mathbb{Q}}\left[\left|\Delta\overline{X}^\delta_T\right|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T|\geq M^\delta\}}\right]\leq\mathbb{E}_{\mathbb{Q}}\left[\left|1-\overline{\mathfrak{R}}_\delta(\overline{\mathcal{X}}^\delta_T)\exp(-\Delta\overline{\mathcal{X}}^\delta_T)\right|\,|\Delta\overline{X}^\delta_T|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T|\geq M^\delta\}}\right],$$

where $M^\delta=N/2\vee(|\Delta\overline{\xi}_\delta|+1)$ The previous inequality, combined with the claim in the last paragraph, yields

$$\limsup_{\delta\downarrow 0}\,(f(\delta)+g(\delta))^{-2}\cdot\mathbb{E}_{\mathbb{Q}}\left[|\Delta\overline{X}^\delta_T|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T|\geq M^\delta\}}\right]<\infty.$$

Now (2.19) follows after noticing $\mathbb{E}_{\mathbb{Q}}\left[|\Delta\overline{X}^\delta_T|\mathbb{I}_{\{|\Delta\overline{X}^\delta_T|\leq M^\delta\}}\right]\leq\mathbb{E}_{\mathbb{Q}}\left[|\Delta\overline{X}^\delta_T|\mathbb{I}_{\{|\Delta\overline{X}^\delta_T|\geq M^\delta\}}\right]$.

Finally, come back to the problem before changing of variable,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[|\Delta X^\delta_T|\right]&\leq\frac{1}{\alpha_\delta}\mathbb{E}_{\mathbb{Q}}\left[|\Delta\overline{X}^\delta_T|\right]+\frac{|\alpha_\delta-1|}{\alpha_\delta}\mathbb{E}_{\mathbb{Q}}[|X^0_T|]\\&\leq C\left[(f(\delta)+g(\delta))^2+g(\delta)\right]\\&\leq C\left(f(\delta)^2+g(\delta)\right),\quad\text{for sufficiently small }\delta.\end{aligned}$$

□

Let us now prove implications of Theorem 1.8 on utility-based pricing.

Proof of Corollary 1.12. Following the change of variable after Lemma 2.1, we can assume without loss of generality that $\alpha_\delta = 1$ for all $\delta \geq 0$ in this proof. Let us first prove

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\left|\frac{d\mathbb{Q}^\delta}{d\mathbb{Q}} - 1\right|\right] = 0. \tag{2.20}$$

The density of $\mathbb{Q}^\delta$ with respect to $\mathbb{Q}$ can be read from Proposition 1.6:

$$\frac{d\mathbb{Q}^\delta}{d\mathbb{Q}} = \frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\,\frac{\mathbb{E}_{\mathbb{P}}[U_0'(\mathcal{X}_T^0)]}{\mathbb{E}_{\mathbb{P}}[U_\delta'(\mathcal{X}_T^\delta)]}. \tag{2.21}$$

We will prove $\mathbb{Q}-\lim_{\delta\downarrow 0} d\mathbb{Q}^\delta/d\mathbb{Q} = 1$. Then (2.20) holds in virtual by Scheffe's lemma.

To prove the convergence in probability, we will prove both factors on the right side of (2.21) converge to 1. It follows from (2.3) that $\mathbb{Q}-\lim_{\delta\downarrow 0}\exp(-\Delta\mathcal{X}_T^\delta) = 1$. On the other hand, for any given N and ϵ, there exists a sufficiently small δ such that $|\mathfrak{R}_\delta(x) - 1| \leq \epsilon$ for $|x| \leq N$. Then $\mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon, |\mathcal{X}_T^\delta| \leq N) = 0$ for sufficiently small δ. Hence $\limsup_{\delta\downarrow 0}\mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon) \leq \limsup_{\delta\downarrow 0}\mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon, |\mathcal{X}_T^\delta| \leq N) + \limsup_{\delta\downarrow 0}\mathbb{Q}(|\mathcal{X}_T^\delta| > N) \leq \mathbb{Q}(|\mathcal{X}_T^0| > N)$, which can be made arbitrarily small for sufficiently large N. Therefore $\mathbb{Q}-\lim_{\delta\downarrow 0}\mathfrak{R}_\delta(\mathcal{X}_T^\delta) = 1$, which combined with the previous convergence implies

$$\mathbb{Q}-\lim_{\delta\downarrow 0}\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)} = \mathbb{Q}-\lim_{\delta\downarrow 0}\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta) = 1.$$

In this paragraph, we will prove

$$\lim_{\delta\downarrow 0}\frac{\mathbb{E}_{\mathbb{P}}[U_\delta'(\mathcal{X}_T^\delta)]}{\mathbb{E}_{\mathbb{P}}[U_0'(\mathcal{X}_T^0)]} = 1.$$

Changing the measure to $\mathbb{Q}$, the previous convergence is equivalent to

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\right] = 1, \tag{2.22}$$

which we will prove next. For any ϵ and N, there exists a sufficiently small δ such that $|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \leq \epsilon$ when $|\mathcal{X}_T^\delta| \leq N$. The previous inequality combined with (2.11) yield

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \\
&\leq (1+\epsilon)\,\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \\
&\to (1+\epsilon)\,\mathbb{Q}(|\mathcal{X}_T^0| \leq N), \quad \text{as } \delta\downarrow 0.
\end{aligned}$$

The previous inequality implies $\limsup_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[U_\delta'(\mathcal{X}_T^\delta)/U_0'(\mathcal{X}_T^0)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \leq 1+\epsilon$. Similar argument also gives $\liminf_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[U_\delta'(\mathcal{X}_T^\delta)/U_0'(\mathcal{X}_T^0)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \geq 1-\epsilon$. On the other hand, it follows from (2.13) that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| \geq N\}}\right] &\leq \overline{\mathfrak{R}}\,\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{|\mathcal{X}_T^\delta| > N\}}\right] \\
&\to \overline{\mathfrak{R}}\,\mathbb{Q}(|\mathcal{X}_T^0| > N), \quad \text{as } \delta\downarrow 0.
\end{aligned}$$

Combining the previous two convergence and sending $N \uparrow \infty$ then $\epsilon \downarrow 0$, we confirm (2.22). Estimates in the previous two paragraphs confirm $\mathbb{Q} - \lim_{\delta\downarrow 0} d\mathbb{Q}^\delta/d\mathbb{Q} = 1$, hence (2.20).

Now switch our attention back to pricing. Since B can be super hedged, for any $\epsilon > 0$, there exists sufficiently large N such that $\mathbb{E}_{\mathbb{Q}^\delta}[B\,\mathbb{I}_{\{B\geq N\}}] \leq \epsilon$ for any $\delta \geq 0$. Indeed,

$$\mathbb{E}_{\mathbb{Q}^\delta}\left[B\,\mathbb{I}_{\{B\geq N\}}\right] \leq \widetilde{x}\,\mathbb{Q}^\delta(B \geq N) + \mathbb{E}_{\mathbb{Q}^\delta}[(G\cdot S)_T] \leq \widetilde{x}\,\mathbb{Q}^\delta(B \geq N) \leq \widetilde{x}\,\mathbb{E}_{\mathbb{Q}^\delta}[B]/N \leq \widetilde{x}^2/N,$$

which is less than ϵ for sufficiently large N. The $\mathbb{Q}^\delta$-supermartingale property of $G\cdot S$ is utilized in the second and fourth inequality. Finally, the previous estimate and (2.20) combined imply

$$\begin{aligned}\left|\mathbb{E}_{\mathbb{Q}^\delta}[B] - \mathbb{E}_{\mathbb{Q}}[B]\right| &\leq \mathbb{E}_{\mathbb{Q}}\left[\left|\frac{d\mathbb{Q}^\delta}{d\mathbb{Q}} - 1\right| B\,\mathbb{I}_{\{B\leq N\}}\right] + \mathbb{E}_{\mathbb{Q}^\delta}[B\,\mathbb{I}_{\{B\geq N\}}] + \mathbb{E}_{\mathbb{Q}}[B\,\mathbb{I}_{\{B\geq N\}}]\\ &\leq N\,\mathbb{E}_{\mathbb{Q}}\left[\left|\frac{d\mathbb{Q}^\delta}{d\mathbb{P}} - 1\right|\right] + 2\epsilon.\end{aligned}$$

Sending $\delta \downarrow 0$ then $\epsilon \downarrow 0$ in the previous inequality, we verify the statement. □

Proof of Corollary 1.13. For any $\mathbb{Q} \in \mathcal{M}^a$, (1.2) implies that $x \leq \mathbb{E}_{\mathbb{Q}}[B] \leq \widetilde{x}$. It then follows from (Owen and Žitković, 2009, Proposition 7.2 (i)) that $x \leq p_\delta \leq \widetilde{x}$ for any $\delta \geq 0$. Therefore in every subsequence of $(p_\delta)_{\delta\geq 0}$ there exists a further subsequence $(p_{\delta_n})_{n\geq 0}$ converging to some limit, say $\widetilde{p}_0$. In the next paragraph, we will prove $\widetilde{p}_0 = p_0$. This implies that the entire sequence of $(p_\delta)_{\delta\geq 0}$ converges to p_0 as well, since the choice of subsequence is arbitrary.

For the subsequence $(\delta_n)_{n\geq 0}$, Assumption 1.7 holds for $\xi_n = x_0 + B - p_{\delta_n}$ and $\xi_0 = x_0 + B - \widetilde{p}_0$ if either $\alpha_\delta = 1$ for any δ or B is bounded. It then follows from Theorem 1.8 ii) that

$$\lim_{\delta_n\downarrow 0} u_{\delta_n}(x_0 + B - p_{\delta_n}) = u_0(x_0 + B - \widetilde{p}_0).$$

Apply Theorem 1.8 ii) when $\xi_n = x_0$,

$$\lim_{\delta_n\downarrow 0} u_{\delta_n}(x_0) = u_0(x_0).$$

Since $u_{\delta_n}(x_0 + B - p_{\delta_n}) = u_{\delta_n}(x_0)$, the previous two convergence combined imply $u_0(x_0 + B - \widetilde{p}_0) = u_0(x_0)$. Then $p_0 = \widetilde{p}_0$ follows from the uniqueness of the indifference price p_0. □

3. Stability for utilities defined on $\mathbb{R}_+$

We will prove Theorem 1.19 in this section. To this end, we can assume without loss of generality that $D_T = 1$ $\mathbb{P}$-a.s.. Otherwise, we can define $\mathbb{P}_D \sim \mathbb{P}$ via $d\mathbb{P}_D/d\mathbb{P} = D_T/\mathbb{E}_{\mathbb{P}}[D_T]$ and work with $\mathbb{P}_D$ instead of $\mathbb{P}$ throughout this section. Assumptions 1.5, 1.15, and 1.18 are enforced throughout this section, (1.5) is satisfied as well. To simplify notation, denote $\widetilde{U}_p(x) = x^p/p$ and $\widetilde{X}^{(p)}$, $\widetilde{Y}^{(p)}$, and $\widetilde{y}_p$ quantities in Proposition 1.17 when U_p is chosen as $\widetilde{U}_p$ in (1.3).

Denote the ratio of optimal wealth processes as

$$r^{(p)} = \frac{X^{(p)}}{\widetilde{X}^{(p)}}$$

and introduce a sequence of auxiliary probability measures $(\mathbb{P}_p)_{p<0}$ via

$$\frac{d\mathbb{P}_p}{d\mathbb{P}} = \frac{\left(\widetilde{X}_T^{(p)}\right)^p}{\mathbb{E}_{\mathbb{P}}\left[\left(\widetilde{X}_T^{(p)}\right)^p\right]}, \qquad \text{for each } p < 0.$$

It follows from Proposition 1.17 that $(X_T^{(p)})^p > 0$, $\mathbb{P}$-a.s., therefore $\mathbb{P}_p \sim \mathbb{P}$ for each $p < 0$. This sequence of auxiliary measures will facilitate various estimates in this section. Another important observation is that $\widetilde{X}^{(p)}$ has the *numéraire property* under $\mathbb{P}_p$, i.e., $\mathbb{E}_{\mathbb{P}_p}[X_T/\widetilde{X}_T^{(p)}] \leq 1$ for any admissible wealth process X. Indeed, Proposition 1.17 implies $\mathbb{E}_{\mathbb{P}}\left[(\widetilde{X}_T^{(p)})^{p-1}(X_T - \widetilde{X}_T^{(p)})\right] \leq 0$ for any admissible X. The claim then follows from changing the measure to $\mathbb{P}_p$ in the previous inequality. As a result, every admissible wealth process X deflated by $\widetilde{X}^{(p)}$ is a $\mathbb{P}_p$-supermartingale; see (Guasoni et al., 2011, Equation (3.10)). In particular, $r^{(p)}$ is a $\mathbb{P}_p$-supermartingale.

As the last section, we start our analysis with the following estimate.

Lemma 3.1. *It holds that*

$$\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[|p|\left|\mathfrak{R}(X_T^{(p)})(r_T^{(p)})^{p-1} - 1\right|\left|1 - r_T^{(p)}\right|\right] = 0.$$

Proof. Throughout this proof we omit the superscript (p) in $X^{(p)}$, $\widetilde{X}^{(p)}$, and $r^{(p)}$ to simplify notation. Applying Proposition 1.17 to U_p and $\widetilde{U}_p$, respectively, yields

$$\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)(\widetilde{X}_T - X_T)\right] \leq 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\left[\widetilde{X}_T^{p-1}(X_T - \widetilde{X}_T)\right] \leq 0.$$

Summing up the previous two inequalities and changing to the measure $\mathbb{P}_p$, we obtain

$$\mathbb{E}_{\mathbb{P}_p}\left[\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right] \leq 0.$$

Similar to Lemma 2.2, we are going to obtain the upper bound of $\mathbb{P}_p$-expectation of $|(U_p'(X_T)\widetilde{X}_T^{1-p} - 1)(1 - X_T/\widetilde{X}_T)|$ by first estimating the upper bound of $\mathbb{P}_p$-expectation of the negative part for $(U_p'(X_T)\widetilde{X}_T^{1-p} - 1)(1 - X_T/\widetilde{X}_T)$. Observe that $(U_p'(X_T)\widetilde{X}_T^{1-p} - 1)(1 - X_T/\widetilde{X}_T) \leq 0$ only when $I_p(\widetilde{X}_T^{p-1}) \leq X_T \leq \widetilde{X}_T$ or $\widetilde{X}_T \leq X_T \leq I_p(\widetilde{X}^{p-1})$, where $I_p = (U_p')^{-1}$. In either cases,

$$\left(\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right)_- \leq \left(1 - \frac{U_p'(\widetilde{X}_T)}{\widetilde{X}_T^{p-1}}\right)\left(1 - \frac{I_p(\widetilde{X}_T^{p-1})}{\widetilde{X}_T}\right).$$

Therefore,

$$\mathbb{E}_{\mathbb{P}_p}\left[\left|\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right|\right] \leq 2\,\mathbb{E}_{\mathbb{P}_p}\left[\left(1 - \mathfrak{R}(\widetilde{X}_T)\right)\left(1 - \frac{I_p(\widetilde{X}_T^{p-1})}{\widetilde{X}_T}\right)\right].$$

Note that

$$\frac{I_p(x^{p-1})}{x} = \frac{I_p(y)}{y^{\frac{1}{p-1}}} = \left(\frac{I_p(y)^{p-1}}{U_p'(I_p(y))}\right)^{\frac{1}{p-1}} = \mathfrak{R}_p(I_p(y))^{\frac{1}{1-p}},$$

where $y = x^{p-1}$. Utilizing the previous identity on $I_p(x^{p-1})/x$, we obtain from the previous inequality and Assumption 1.15 that

$$\mathbb{E}_{\mathbb{P}_p}\left[\left|\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}}-1\right)\left(1-\frac{X_T}{\widetilde{X}_T}\right)\right|\right] \leq 2\max\left\{(\overline{\mathfrak{R}}_p-1)(\overline{\mathfrak{R}}_p^{\frac{1}{1-p}}-1), (1-\underline{\mathfrak{R}}_p)(1-\underline{\mathfrak{R}}_p^{\frac{1}{1-p}})\right\}.$$

It follows from (1.5) that $\limsup_{p\downarrow-\infty}|p|(\overline{\mathfrak{R}}_p-1)<\infty$ and $\lim_{p\downarrow-\infty}\overline{\mathfrak{R}}_p^{\frac{1}{1-p}}=\lim_{p\downarrow-\infty}\exp(\frac{1}{1-p}\log\overline{\mathfrak{R}}_p)=1$. Therefore the first term on the right side of the previous inequality, after multiplying by $|p|$, converges to 0 as $p\downarrow-\infty$. Similar argument applies to the second term as well. As a result, the left side expectation, after multiplying $|p|$, converges to 0 as $p\downarrow-\infty$. □

The previous estimate induces the convergence of $r_T^{(p)}$ in the following sense.

Corollary 3.2. *It holds that*

$$\lim_{p\downarrow-\infty}\mathbb{P}_p\left(\left|(r_T^{(p)})^p-1\right|\geq\epsilon\right)=0, \quad \text{for any } \epsilon>0.$$

Proof. Throughout this proof we still omit the superscript (p). When $r_T^p\geq 1+\epsilon$, $1-r_T\geq 1-(1+\epsilon)^{1/p}$. Note that $(1+\epsilon)^{1/p}=\exp(p^{-1}\log(1+\epsilon))=1+p^{-1}\log(1+\epsilon)+o(p^{-1})$. Hence $\lim_{p\downarrow-\infty}-p(1-(1+\epsilon)^{1/p})=\log(1+\epsilon)>0$. Therefore when $r_T^p\geq 1+\epsilon$, $-p(1-r_T)\geq\frac{1}{2}\log(1+\epsilon)>0$ for sufficiently small p. When $r_T^p\leq 1-\epsilon$, we can similarly obtain $-p(r_T-1)\geq-\frac{1}{2}\log(1-\epsilon)>0$ for sufficiently small p. Set $\eta=\min\{\frac{1}{2}\log(1+\epsilon),-\frac{1}{2}\log(1-\epsilon)\}>0$. The previous two estimates combined yield

$$-p\,|r_T-1|\geq\eta \quad \text{when } |r_T^p-1|\geq\epsilon \text{ for sufficiently small } p.$$

On the other hand, when $r_T^p\geq 1+\epsilon$, $r_T^{p-1}\geq 1+\epsilon/2$ for sufficiently small p. Moreover (1.5) and Assumption 1.15 combined imply that $\mathfrak{R}_p(X_T)\geq\underline{\mathfrak{R}}_p\geq(1+\epsilon/2)^{-\frac{1}{2}}$ for sufficiently small p. As a result,

$$\mathfrak{R}_p(X_T)r_T^{p-1}-1\geq(1+\epsilon/2)^{-\frac{1}{2}}(1+\epsilon/2)-1=(1+\epsilon/2)^{\frac{1}{2}}-1>0, \quad \text{when } r_T^p-1\geq\epsilon \text{ for sufficiently small } p.$$

Similarly,

$$1-\mathfrak{R}(X_T)r_T^{p-1}\geq 1-(1-\epsilon/2)^{\frac{1}{2}}>0, \quad \text{when } r_T^p-1\leq-\epsilon, \text{ for sufficiently small } p.$$

Combining estimates in the last two paragraphs, we obtain

$$|p|\left|\mathfrak{R}(X_T)r_T^{p-1}-1\right||1-r_T|\geq\eta\cdot\min\left\{(1+\epsilon/2)^{\frac{1}{2}}-1,1-(1-\epsilon/2)^{\frac{1}{2}}\right\}>0, \quad \text{when } |r_T^p-1|\geq\epsilon,$$

for for sufficiently small p. The statement then follows from the previous inequality and Lemma 3.1. □

The previous convergence in probability implies that $(r_T^{(p)})^p$ converges to 1 in expectation.

Proposition 3.3. *It holds that*

$$\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}_p}\left[\left|\left(r_T^{(p)}\right)^p-1\right|\right]=0.$$

Proof. Throughout this proof we omit the superscript (p). The proof is split into two steps. The first step proves

$$\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}[r_T^p] = 1. \tag{3.1}$$

The second step confirms the statement.

<u>*Step 1:*</u> After the measure $\mathbb{P}_p$ is changed to $\mathbb{P}$, (3.1) is equivalent to

$$\lim_{p\downarrow-\infty} \frac{\mathbb{E}_{\mathbb{P}}[X_T^p]}{\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]} = 1, \tag{3.2}$$

which we will prove in this step. We have seen in Proposition 1.17 that

$$\frac{\underline{\mathfrak{R}}_p}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p] \leq x_0 = \frac{1}{y_p}\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)X_T\right] \leq \frac{\overline{\mathfrak{R}}_p}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p],$$

where Assumption 1.15 is used to obtain two inequalities. Sending $p \downarrow -\infty$ in the previous inequality, we obtain from $\underline{\mathfrak{R}}_p, \overline{\mathfrak{R}}_p \to 1$,

$$\lim_{p\downarrow-\infty} \frac{1}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p] = x_0.$$

The optimality of $\widetilde{X}$ gives $\mathbb{E}_{\mathbb{P}}[X_T^p]/p \leq \mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]/p = x_0\widetilde{y}_p/p$. The previous convergence then yields

$$\limsup_{p\downarrow-\infty} \frac{\widetilde{y}_p}{y_p} \leq 1.$$

The reverse inequality on the limit inferior will be proved in the next paragraph.

Note that $\frac{y}{I_p(y)^{p-1}} = \frac{U_p'(x)}{x^{p-1}}$ for $x = I_p(y)$. Then Assumption 1.15 gives $\underline{\mathfrak{R}}_p \leq \frac{y}{I_p(y)^{p-1}} \leq \overline{\mathfrak{R}}_p$, which yields

$$\underline{\mathfrak{R}}_p^{\frac{1}{1-p}} \leq \frac{I_p(y)}{y^{\frac{1}{p-1}}} \leq \overline{\mathfrak{R}}_p^{\frac{1}{1-p}}, \qquad \text{for } y > 0.$$

Here both upper and lower bounds in the previous inequality converge to 1 as $p \downarrow -\infty$ owing to (1.5). Proposition 1.17 then yields

$$x_0 = \mathbb{E}_{\mathbb{P}}\left[Y_T I_p\left(y_p Y_T\right)\right] \leq \overline{\mathfrak{R}}^{\frac{1}{1-p}}\mathbb{E}_{\mathbb{P}}\left[Y_T\left(y_p Y_T\right)^{\frac{1}{p-1}}\right] = \overline{\mathfrak{R}}^{\frac{1}{1-p}} y_p^{\frac{1}{p-1}}\mathbb{E}_{\mathbb{P}}\left[Y_T^q\right],$$

where $q := p/p-1$. Note $\mathbb{E}_{\mathbb{P}}[Y_T^q]^{1-p} \leq \mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p/x_0^p]$ follows from $\mathbb{E}_{\mathbb{P}}[Y_T\widetilde{X}_T/x_0] \leq 1$ and Hölder's inequality (see e.g. (Guasoni and Robertson, 2012, Lemma 5)). The previous two inequalities combined yield $x_0 y_p \leq \overline{\mathfrak{R}}_p\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p] = \overline{\mathfrak{R}}_p\, x_0\widetilde{y}_p$. Sending $p \downarrow -\infty$ and utilizing $\lim_{p\downarrow-\infty}\overline{\mathfrak{R}}_p = 1$, we obtain from the previous inequality

$$\liminf_{p\downarrow-\infty} \frac{\widetilde{y}_p}{y_p} \geq 1.$$

Estimates from the last two paragraphs yield $\lim_{p\downarrow-\infty} y_p/\widetilde{y}_p = 1$, which is equivalent to

$$\lim_{p\downarrow-\infty} \frac{\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)X_T\right]}{\mathbb{E}_{\mathbb{P}}\left[\widetilde{X}_T^p\right]} = 1.$$

Since $\underline{\mathfrak{R}}_p\mathbb{E}_{\mathbb{P}}[X_T^p] \leq \mathbb{E}_{\mathbb{P}}[U_p'(X_T)X_T] \leq \overline{\mathfrak{R}}_p\mathbb{E}_{\mathbb{P}}[X_T^p]$, (3.2) follows from dividing by $\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]$ on both sides of the previous inequality and sending $p \downarrow -\infty$.

Step 2: Fix $N > 1$. Let us prove in this paragraph $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p \le N\}}\right] = 0$. To this end, for any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^P \le N\}}\right] &= \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^P \le N, |r_T^p - 1| \le \epsilon\}}\right] + \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^P \le N, |r_T^p - 1| > \epsilon\}}\right] \\
&\le \epsilon + (N-1)\,\mathbb{P}_p(|r_T^p - 1| > \epsilon) \\
&\to \epsilon, \qquad \text{as } p \downarrow -\infty,
\end{aligned}$$

where the convergence follows from Corollary 3.2. Therefore the claim is confirmed since the choice of ϵ is arbitrary in the previous inequality.

Let us prove $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p > N\}}\right] = 0$ in this paragraph. Combining this convergence and the one in the last paragraph confirms $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\right] = 0$. To prove the desired claim,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p > N\}}\right] &\le \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\,\mathbb{I}_{\{r_T^p > N\}}\right] \\
&= \mathbb{E}_{\mathbb{P}_p}[r_T^p] - \mathbb{E}_{\mathbb{P}_p}\left[(r_T^p - 1)\,\mathbb{I}_{\{r_T^p \le N\}}\right] - \mathbb{P}_p(r_T^p \le N) \\
&\to 1 - 0 - 1 = 0, \qquad \text{as } p \downarrow -\infty,
\end{aligned}$$

where the convergence of three terms follow from the result in Step 1, the result in the last paragraph, and Corollary 3.2, respectively. □

The convergence of optimal payoffs in Proposition 3.3 implies the ratio of optimal wealth processes converges uniformly in probability. The proof of the following two results adapt argument in (Kardaras, 2010, Theorem 2.5) into our context.

Corollary 3.4. *It holds that*

$$\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\sup_{t\in[0,T]} \left|(r_T^{(p)})^p - 1\right| \ge \epsilon\right) = 0.$$

Proof. The superscript (p) is omitted throughout to simplify notation. Recall that r is a $\mathbb{P}_p$-supermartingale; see the discussion before Lemma 3.1. Then $p < 0$ implies that r^p is a $\mathbb{P}_p$-submartingale. Indeed,

$$\mathbb{E}_{\mathbb{P}_p}\left[r_t^p \,|\, \mathcal{F}_s\right] \ge \left(\mathbb{E}_{\mathbb{P}_p}\left[r_t \,|\, \mathcal{F}_s\right]\right)^p \ge r_s^p, \qquad \text{for any } s \le t,$$

where the Jensen's inequality is applied to obtain the first inequality.

In the next two paragraphs, we will prove

$$\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\left|\sup_{t\in[0,T]} r_t^p - 1\right| \ge \epsilon\right) = 0 \quad \text{and} \quad \lim_{p\downarrow-\infty} \mathbb{P}_p\left(\left|\inf_{t\in[0,T]} r_t^p - 1\right| \ge \epsilon\right) = 0, \tag{3.3}$$

for any fixed $\epsilon > 0$. These two convergence combined confirm the statement.

To prove the first convergence in (3.3), define $\tau_p := \inf\{t \ge 0 \,|\, r_t^p \ge 1 + \delta\} \wedge T$ for $p < 0$ and $\delta > 0$. It then suffices to prove

$$\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\tau_p < T\right) = 0,$$

since δ is arbitrarily chosen. Suppose the previous convergence does not hold. Then there exists $\eta > 0$ and a subsequence, which we still denote by τ_p, such that $\lim_{p\downarrow-\infty} \mathbb{P}_p(\tau_p < T) = \eta$. It then follows from Proposition 3.3 that

$$\left|\mathbb{E}_{\mathbb{P}_p}\left[r_T^p\,\mathbb{I}_{\{\tau_p=T\}}\right] - \mathbb{P}_p(\tau_p = T)\right| = \left|\mathbb{E}_{\mathbb{P}_p}\left[(r_T^p - 1)\,\mathbb{I}_{\{\tau_p=T\}}\right]\right| \le \mathbb{E}_{\mathbb{P}_p}[|r_T^p - 1|] \to 0, \quad \text{as } p \downarrow -\infty.$$

This implies $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\,\mathbb{I}_{\{\tau_p=T\}}\right] = 1-\eta$. On the other hand, the $\mathbb{P}_p$-submartingale property of r^p implies

$$1 \le \mathbb{E}_{\mathbb{P}_p}[r_{\tau_p}^p] \le \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\right] \to 1, \quad \text{as } p \downarrow -\infty,$$

where the last convergence follows from (3.1). Hence $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\right] = 1$. Therefore

$$\begin{aligned}1 = \lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}[r_{\tau_p}^p] &\ge \liminf_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\,\mathbb{I}_{\{\tau_p<T\}}\right] + \lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\,\mathbb{I}_{\{\tau_p=T\}}\right]\\ &\ge (1+\delta)\eta + (1-\eta) = 1+\delta\eta > 1,\end{aligned}$$

which is a contradiction.

The proof of the second convergence in (3.3) is similar. Redefine $\tau_p = \inf\{t \ge 0 \,|\, r_t^p \le 1-\delta\} \wedge T$. Suppose that $\lim_{p\downarrow-\infty} \mathbb{P}_p(\tau_p < T) = 0$ does not hold, then there exists a subsequence, which we still denote by τ_p, such that $\lim_{p\downarrow-\infty} \mathbb{P}_p(\tau_p < T) = \eta$ for some $\eta > 0$. The same argument as that in the last paragraph yields $\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\,\mathbb{I}_{\{\tau_p=T\}}\right] = 1-\eta$. Then

$$\begin{aligned}1 = \lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}[r_{\tau_p}^p] &\le \limsup_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\,\mathbb{I}_{\{\tau_p<T\}}\right] + \lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\,\mathbb{I}_{\{\tau_p=T\}}\right]\\ &\le (1-\delta)\eta + (1-\eta) = 1-\delta\eta < 1,\end{aligned}$$

which is again a contradiction. □

Our next goal is to pass from the convergence of optimal wealth processes to the convergence of optimal strategies.

Proposition 3.5. *If S is continuous, then the following statements hold for any $\epsilon > 0$:*

i) $\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\left[\left(r^{(p)}\right)^p, \left(r^{(p)}\right)^p\right]_T \ge \epsilon\right) = 0$;

ii) $\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\left[\mathcal{L}^{(p)}, \mathcal{L}^{(p)}\right]_T \ge \epsilon\right) = 0$, *where* $\mathcal{L}^{(p)} := \int_0^{\cdot} \left(1/(r_t^{(p)})^p\right) d(r_t^{(p)})^p$, *i.e.,* $\mathcal{L}^{(p)}$ *is the stochastic logarithm of* $(r^{(p)})^p$.

Remark 3.6. Under the structure condition, $[\mathcal{L}^{(p)}, \mathcal{L}^{(p)}]_T = \int_0^T p(\pi_p - \widetilde{\pi}_p)_t\, d\langle M\rangle_t\, p(\pi_p - \widetilde{\pi}_p)_t$, which measures how far $p(\pi_p - \widetilde{\pi}_p)$ is away from 0.

Proof. The superscript (p) on r and $\mathcal{L}$ is omitted throughout this proof. Note that $[r^p, r^p]_\cdot = \int_0^\cdot |r^p|^2\, d[\mathcal{L}, \mathcal{L}]_t$. Statement ii) then follows from statement i) and Corollary 3.2 directly. We will prove statement i) in what follows.

Define $\tau_p = \inf\{t \ge 0 \,|\, r_t^p \ge 2\} \wedge T$. It follows from Corollary 3.4 that $\lim_{p\downarrow-\infty} \mathbb{P}_p(\tau_p = T) = 1$. Therefore it suffices to prove

$$\lim_{p\downarrow-\infty} \mathbb{P}_p\left([r^p, r^p]_{T\wedge\tau_p} \ge \epsilon\right) = 0. \tag{3.4}$$

Set $Z^{(p)}_{\cdot} = r^p_{\cdot\wedge\tau_p}$. Since r^p is a $\mathbb{P}_p$-submartingale, so is $Z^{(p)}$. Therefore (3.1) induces $\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}_p}[Z^{(p)}_T] = 1$. On the other hand, the continuity of S implies the continuity of r^p, hence $Z^{(p)}$ is bounded from above by 2 for all $p<0$. The Doob-Meyer decomposition gives $Z^{(p)} = M^{(p)} + B^{(p)}$ where $M^{(p)}$ is a $\mathbb{P}_p$-martingale and $B^{(p)}$ is a continuous nondecreasing process with $B^{(p)}_0 = 0$. The continuity of $B^{(p)}$ follows from (Karatzas and Shreve, 1991, Theorem 1.4.14). Note $sup_{t\in[0,T]}|Z^{(p)}_t - 1| \le \sup_{t\in[0,T]}|M^{(p)}_t - 1| + B^{(p)}_T$. Hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|M^{(p)}_t - 1|\right] &\le \mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z^{(p)}_t - 1|\right] + \mathbb{E}_{\mathbb{P}_p}[B^{(p)}_T] \\
&= \mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z^{(p)}_t - 1|\right] + \mathbb{E}_{\mathbb{P}_p}[Z^{(p)}_T] - \mathbb{E}_{\mathbb{P}_p}[M^{(p)}_T] \\
&\to 0 + 1 - 1 = 0, \quad \text{ as } p\downarrow -\infty,
\end{aligned}$$

where $\mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z^{(p)}_t - 1|\right] \to 0$ holds owing to $|Z^{(p)} - 1| \le 1$ and Corollary 3.4, $\mathbb{E}_{\mathbb{P}_p}[M^{(p)}_T] = 1$ holds because $M^{(p)}$ is a $\mathbb{P}_p$-martingale. Therefore the Davis inequality yields $\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}_p}[[M^{(p)}, M^{(p)}]^{1/2}_T] = 0$, which implies $\lim_{p\downarrow-\infty}\mathbb{P}_p([M^{(p)}, M^{(p)}]_T \ge \epsilon) = 0$. Hence (3.4) is confirmed, since $B^{(p)}$ is a continuous increasing process.

□

Last step to prove Theorem 1.19, we are going to identify limit of $\mathbb{P}_p$ as $p\downarrow-\infty$. To this end, we recall the opportunity process for power utility. The càdlàg semimartingale $L^{(p)}$ is called the *opportunity process* for the power utility x^p/p if it satisfies

$$L^{(p)}_t \frac{1}{p}(X_t(\pi))^p = \text{esssup}_{\widetilde{\pi}\in\mathcal{A}(\pi)}\mathbb{E}_{\mathbb{P}}\left[\frac{1}{p}(X(\widetilde{\pi})_T)^p \middle| \mathcal{F}_t\right], \quad 0\le t\le T,$$

for all $\pi\in\mathcal{A}$, where $\mathcal{A}(\pi) = \{\widetilde{\pi}\in\mathcal{A} : \widetilde{\pi} = \pi \text{ on } [0,t]\}$. The existence and uniqueness of $L^{(p)}$ have been proved in (Nutz, 2010, Proposition 3.1). Thanks to the scaling property of power utility, $L^{(p)}$ can be viewed as a dynamic version of the reduced value function. In particular, the definition above implies that $L^{(p)}_0 x_0^p/p = \widetilde{u}_p(x_0)$, where $\widetilde{u}_p(x_0)$ is defined in (1.3) with $U_p(x) = x^p/p$, and $L^{(p)}_0 x_0^{p-1} = \widetilde{y}_p = \widetilde{u}'_p(x_0)$. As a result, the density of $\mathbb{P}_p$ can be rewritten as

$$\frac{d\mathbb{P}_p}{d\mathbb{P}} = \frac{\left(\widetilde{y}_p\widetilde{Y}^{(p)}_T\right)^q}{p\widetilde{u}_p(x_0)} = \frac{\left(L^{(p)}_0\widetilde{Y}^{(p)}_T\right)^q}{L^{(p)}_0} = \frac{\left(\widetilde{Y}^{(p)}_T\right)^q}{\left(L^{(p)}_0\right)^{1-q}},$$

where $q = p/(p-1)$. As $p\downarrow-\infty$, using convergence results in Nutz (2012), we will show that the denominator in the rightmost equality above converges to 1 and the numerator converges to the density of the minimal entropy measure $\mathbb{Q}$. Therefore convergence under the sequence of measures $(\mathbb{P}_p)_{p<0}$ in Proposition 3.5 can be replaced by convergence in probability $\mathbb{Q}$. This, combined with (Nutz, 2012, Theorem 3.2), concludes the proof of Theorem 1.19.

Proof of Theorem 1.19. Let us first prove

$$\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}}\left[\left|\frac{d\mathbb{P}_p}{d\mathbb{P}} - \frac{d\mathbb{Q}}{d\mathbb{P}}\right|\right] = 0. \tag{3.5}$$

To this end, when S is continuous, it follows from (Nutz, 2012, Theorem 6.6) that $\lim_{p\downarrow-\infty} L_0^{(p)} = L_0^{\exp}$, where $L^{\exp}$ is the opportunity process for exponential utility $-\exp(-x)$ defined in the similar fashion as that for power utility; see (Nutz, 2012, equation (6.3)). Since $q \to 1$ as $p \downarrow -\infty$, then $\lim_{p\downarrow-\infty}(L_0^{(p)})^{1-q} = 1$. On the other hand, when S is continuous and $L^{(p)}$ is continuous for all $p < 0$, (Nutz, 2012, Proposition 6.13) proved that $\widetilde{Y}^{(p)}$ converges in the semimartingale topology to the density of $\mathbb{Q}$ as $p \downarrow -\infty$. The definition of convergence in semimartingale topology says $\mathbb{P} - \lim_{p\downarrow-\infty} \xi_0^{(p)}\widetilde{Y}_0^{(p)} + \left(\xi^{(p)} \cdot \widetilde{Y}^{(p)}\right)_t = 0$ for any $t \in [0,T]$ and every sequence $(\xi^{(p)})_{p<0}$ of elementary predictable processes with $|\xi^{(p)}| \leq 1$. In particular, choosing $\xi^{(p)} = 1$ for all $p < 0$, we obtain $\mathbb{P} - \lim_{p\downarrow-\infty} \widetilde{Y}_T^{(p)} = d\mathbb{Q}/d\mathbb{P}$. Hence $\mathbb{P} - \lim_{p\downarrow-\infty}(\widetilde{Y}_T^{(p)})^q = d\mathbb{Q}/d\mathbb{P}$, which, after combined with $\lim_{p\downarrow-\infty}(L_0^{(p)})^{1-q} = 1$, implies

$$\mathbb{P} - \lim_{\delta\downarrow-\infty} \frac{d\mathbb{P}_p}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Hence the $\mathbb{L}^1(\mathbb{P})$ convergence in (3.5) follows from the previous convergence in probability $\mathbb{P}$ and Scheffe's lemma. The assumptions on the continuity of S and $L^{(p)}$ for all $p < 0$ are ensured by Assumption 1.18; see (Nutz, 2012, Remark 4.2).

Proposition 3.5 ii) and (3.5) combined yield $\mathbb{Q} - \lim_{p\downarrow-\infty} [p(\pi_p - \widetilde{\pi}_p) \cdot R]_T = 0$, where $[Z] := [Z, Z]$ is the quadratic variation for the semimartingale Z. Hence

$$\mathbb{P} - \lim_{p\downarrow-\infty} [(1-p)(\pi_p - \widetilde{\pi}_p) \cdot R]_T = 0, \tag{3.6}$$

since $\mathbb{Q} \sim \mathbb{P}$. On the other hand, (Nutz, 2012, Theorem 3.2) proved that $(1-p)\widetilde{\pi}_p \to \hat{\vartheta}$ in $L^2_{loc}(M)$ as $p \downarrow -\infty$. This implies $\mathbb{P} - \lim_{p\downarrow-\infty}[((1-p)\widetilde{\pi}_p - \hat{\vartheta}) \cdot R]_{T\wedge\tau_n} = 0$, for a sequence of stopping time (τ_n) with $\lim_{n\uparrow\infty} \tau_n = \infty$; see (Nutz, 2012, Lemma A.3). The previous convergence then yields

$$\mathbb{P} - \lim_{p\downarrow-\infty} \left[((1-p)\widetilde{\pi}_p - \hat{\vartheta}) \cdot R\right]_T = 0. \tag{3.7}$$

Finally, the statement is confirmed via

$$\begin{aligned}\left[((1-p)\pi_p - \hat{\vartheta}) \cdot R\right]_T &= \left[(1-p)(\pi_p - \widetilde{\pi}_p) \cdot R + ((1-p)\widetilde{\pi}_p - \hat{\vartheta}) \cdot R\right]_T \\ &\leq 2\left[(1-p)(\pi_p - \widetilde{\pi}_p) \cdot R\right]_T + 2\left[((1-p)\widetilde{\pi}_p - \hat{\vartheta}) \cdot R\right]_T,\end{aligned}$$

where both terms in the right side converge in probability $\mathbb{P}$ to zero as we have seen in (3.6) and (3.7). $\square$

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